Math 160 Discussion Notes
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2.5 The Gauss-Jordan Method of finding an inverse

Say we have matrix A, and a sequence of Row elementary row operations E_1, E_2, … E_k which will reduce A to I_n. It turns out that the same sequence of row operations will reduce I_n to A^{-1}.

An elementary row operation on an nxn matrix can be represented by an elementary matrix and performed with matrix multiplication. For example, the operation "$R_1 \leftrightarrow R_2$" can be performed by left-multiplying by the matrix $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$. The operation "$2R_1+R_3 = R_3$" is represented by $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$.

So the sequence of row operations E_1, E_2, … E_k on matrix A can be written

$E_1E_2\cdots E_k A = I$ If A^{-1} exists, then we can right-multiply both sides by A^{-1}

$E_1E_2\cdots E_k AA^{-1} = IA^{-1}$

$E_1E_2\cdots E_k I = A^{-1}$

We can use this fact to develop a method to find the inverse of a matrix. To find the inverse of nxn matrix A, we augment with the Identity to form a nx2n matrix [A I]. We perform Gauss-Jordan reduction on the matrix and the result is [I A^{-1}]. If we cannot reduce A to I using row operations, then A has no inverse.

This is the **Gauss-Jordan Method for finding the inverse of a matrix**

ex) Find the inverse of $\boldsymbol{A=\begin{bmatrix}7&3\\5&2\end{bmatrix}}$

We augment the matrix to form $\begin{bmatrix}7&3&1&0\\5&2&0&1\end{bmatrix}$ And perform row operations to reduce the left-side to the identity.

$$\begin{bmatrix}7&3&1&0\\5&2&0&1\end{bmatrix} \quad \underset{\rightarrow}{\frac{1}{7}R_1}\begin{bmatrix}1&\frac{3}{7}&\frac{1}{7}&0\\5&2&0&1\end{bmatrix} \quad \underset{\rightarrow}{-5R_1}\begin{bmatrix}1&3/7&1/7&0\\0&5&0&1\end{bmatrix}$$

$$\underset{\rightarrow}{-7R_2}\begin{bmatrix}1&\frac{3}{7}&\frac{1}{7}&0\\0&1&5&-7\end{bmatrix} \quad \underset{\rightarrow}{\frac{-3}{7}R_2+R_1}\begin{bmatrix}1-0&\frac{3}{7}-\frac{3}{7}&\frac{1}{7}-\frac{15}{7}&0+3\\0&1&5&-7\end{bmatrix}=\begin{bmatrix}1&0&-2&3\\0&1&5&-7\end{bmatrix}$$

So $A^{-1}=\begin{bmatrix}-2&3\\5&-7\end{bmatrix}$

ex) Find the inverse of $\boldsymbol{B=\begin{bmatrix}1&2&-2\\1&1&1\\0&0&1\end{bmatrix}}$

We augment B to form $\begin{bmatrix}1&2&-2&1&0&0\\1&1&1&0&1&0\\0&0&1&0&0&1\end{bmatrix}$ which, after Gauss-Jordan elimination, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -4 \\ 0 & 1 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \text{, so } B^{-1}=\begin{bmatrix} -1 & 2 & -4 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

ex) Find the inverse of $C=\begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 11 & 3 \end{bmatrix}$

The augmented matrix $\begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 11 & 3 & 0 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2/5 & 0 & -11/15 & 2/15 \\ 0 & 1 & 1/5 & 0 & 2/15 & 1/15 \\ 0 & 0 & 0 & 1 & 1/3 & -1/3 \end{bmatrix}$.

Because we have the 3 zeroes in the first 3 columns of the last row, we can say that C has no inverse.

ex) Find a 2x2 matrix D such that $D\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ **and** $D\begin{bmatrix} 5 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

We can consider this problem as matrix D multiplied by 2x2 matrix $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ gives $\begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$

$D\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$. If we can find the inverse of $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ we can express $D=\begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1}$.

$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1}=\dfrac{1}{3(2)-5(1)}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, so

$$D=\begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} -1(3)+0(-1) & -1(-5)+0(2) \\ 4(3)+2(-1) & 4(-5)+2(2) \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 10 & -16 \end{bmatrix}$$