Math 160 , Finite Mathematics for Business

Section 5.5 - Discussion Notes
Brian Powers - TA - Fall 2011

Factorial

In these counting problems we make frequent use of the **factorial** function. When we say "n factorial", we mean the product of all numbers from n to 1. In notation, we write:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

ex) Calculate $4!$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

ex) Calculate $\frac{5!}{3!}$

$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1}} = 5 \cdot 4 = 20$

ex) Simplify $\frac{n!}{(n-2)!}$

We can expand a factorial by any number of terms we want. So $n! = n \cdot (n-1)! = n \cdot (n-1) \cdot (n-2)!$ and so on. Thus we can simplify the fraction:

$\frac{n!}{(n-2)!} = \frac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!} = \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1) = n^2 - n$

Permutations

A permutation is an ordering of things. We often want to know how many different orderings can we make of r things out of a total of n. This is written $P(n, r)$ and is read as "A permutation of r out of n". There are two equivalent formulas for a permutation:

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$$
$$\text{or } P(n, r) = \frac{n!}{(n-r)!}$$

Both of these formulas will calculate the same number, but one may be more convenient depending on the circumstances. To clarify the first formula, we simply take the product of the first r numbers from $n!$

ex) Calculate $P(4, 2)$

$P(4, 2) = 4 \cdot 3 = 12$. Alternatively, you can use the second formula:

$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4 \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{2 \cdot 1}} = 4 \cdot 3 = 12$

ex) Calculate $P(6, 3)$

$P(6, 3) = 6 \cdot 5 \cdot 4 = 120$

ex) If an art gallery has 7 paintings, in how many ways can it arrange 4 of them across the gallery wall?

In this example we are asked how many different orderings of 4 out of the 7 paintings there are. This is a permutation of 4 out of 7 things, and is calculated by $P(7, 4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$. There are 840 ways that 4 of the 7 paintings can be arranged across the wall.

ex) In how many ways can 3 basketball franchises be awarded to 5 cities that have applied for them?

Let's call the basketball franchises Team 1, Team 2 and Team 3. So the question is asking in how many ways can 3 of the cities be awarded team 1, 2 and 3. This is an ordering problem - we want to count the number of permutations of 3 out of 5 cities. This is done in $P(5, 3) = 5 \cdot 4 \cdot 3 = 60$ possible ways.

You may notice that a special case of the permutation problem is if you want to order n out of n things.

$P(n,n) = n(n-1)(n-2)\cdots(3)(2)(1) = n!$.

ex) how many 4-letter "words" can be formed by rearranging the letters M A T H? (These do not have to be real dictionary words)
This ordering problem is asking in how many ways can you order 4 out of 4 letters. This is $P(4,4) = 4! = 24$.

Combinations
The next modification to counting problems is when there is no ordering involved. An unordered selection of r out of n things is called a **combination** and we write $C(n,r)$. Often we will simply say "n choose r" and use $\binom{n}{r}$ as notation. A combination of r out of n things is calculated using either of the following equivalent formulas:

$$C(n,r) = \frac{P(n,r)}{r!} \text{ or alternatively } C(n,r) = \binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}.$$

ex) Calculate $C(12,2)$
$C(12,2) = \frac{P(12,2)}{2!} = \frac{12\cdot 11}{2} = 66$

ex) In how many ways can you pick 2 books from your bookshelf of 9 books to take with you for a weekend trip?
This problem is an unordered counting problem, because the order of the two books you pick does not matter once you've put them in your suitcase. So we use the combination formula to count this, not permutation. So this is a combination of 2 out of 9 things. This can be done in $C(9,2) = \frac{9\cdot 8}{2!} = 36$ possible ways.

ex) If a pizza parlor offers 8 different toppings, how many 3 topping pizzas are possible?
This problem also involved unordered combinations of things. A pizza with peppers, mushrooms and garlic is no different than a pizza with garlic, peppers and mushrooms. So we use the combination formula. Let's start using the "choose" terminology, as it more intuitively describes what we're doing - we are choosing 3 toppings from the 8. That is $\binom{8}{3} = \frac{8!}{3!\cdot 5!} = 56$. So there are 56 possible 3-topping pizzas.

ex) How many 6 digit numbers are strictly decreasing when read left-to-right?
An example of a strictly decreasing 6 digit number is 864,321. Often with these problems the trick is to figure out a way of thinking about it where you can use the tools we've learned. For this problem, first list the digits from 9 down to 0:

$$9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1\ 0$$

Any strictly decreasing number can be formed by removing any given 4 of the numbers from the list - the remaining 6 digits would create a decreasing number. For example, remove 7, 5, 3 and 0:

$$9\ 8\ \cancel{7}\ 6\ \cancel{5}\ 4\ \cancel{3}\ 2\ 1\ \cancel{0}$$

and we get 986,421. So by choosing 4 of the 10 digits to remove, we create a 6 digit decreasing number. So in how many ways can we choose 4 of 10 digits? This is 10 choose 4 $= \binom{10}{4} = \frac{10!}{4!6!} = 210$.

How many 5 card poker hands are possible from a deck of 52 cards?
When considering poker hands, we are not concerned with the order of the cards as they are dealt. Because the order doesn't matter, this is a combination problem. In how many ways can we be dealt 5 out of 52 cards? This is 52 choose 5 $= \binom{52}{5} = 2,598,960$.

How many 4 digit numbers can be formed from the numbers $\{1,2,3,4,5,6,7\}$?
With no additional restrictions, repetition of the same digit is allowed. There are 7 possibilities for the first digit, 7 for the second, 7 for the third and 7 for the fourth. By the multiplication principal, there will be $7\cdot 7\cdot 7\cdot 7 = 7^4 = 2,401$ possible numbers.
What if adjacent digits of our 4 digit number must be different?
In this case, we still have 7 possibilities for the first digit, but only 6 for the second number. The third

number cannot be the same as the second, but because it is not adjacent to the first that digit is now a possibility again. So for the third and fourth digit there are 6 possibilities. Therefore there are $7 \cdot 6 \cdot 6 \cdot 6 = 1,512$ possible numbers that can be formed.

What if you cannot repeat any number?

If no repetitions are allowed, this has become a permutation of 4 out of 7 digits (because the order of the 4 digits you use matters). There are $P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ permutations of 4 digits from the 7.