Math 160 , Finite Mathematics for Business

Section 5.6 - Discussion Notes
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1) In how many ways can 100 students be assigned to dorms A, B and C with 25 assigned to A, 40 to B and 35 to C?

We use the multiplication principal to solve this. We can look at the dorms individually and make this a problem with a sequence of decisions, each with a certain number of ways of making it:
a) which 25 students go to dorm A,
b) which 40 students go to dorm B, and
c) which 35 go to dorm C.
We are not yet looking at which room the students go to within dorm A, B and C, just the building itself. So we are in essence choosing 25 of the 100 total students to go into dorm A. This is $\binom{100}{25}$.
After 25 have been placed into dorm A, there are 75 students who do not have dorm assignments. We need to choose 40 to go into dorm B, which is $\binom{75}{40}$.
Now there are only 35 who do not have dorm assignments, but they all must necessarily go into dorm C - there is no decision to be made, so only 1 possibility exists.
The product of the decisions comes to:
$\binom{100}{25} \cdot \binom{75}{40} \cdot 1 \approx 7.44 \times 10^{44}$

2) If you toss a coin 6 times, how many possible outcomes are there?

For the coin toss problems we consider each distinct outcome as a record of the coin flips in the order they came about. Thus "HHTHTT" is a distinct outcome from "HTTTHH" even though both contained 3 heads. Because each flip of the coin offers two possibilities and we are flipping 6 times, the multiplication principle tells us that there will be:
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$ possible outcomes.

How many outcomes will have exactly 3 heads?

To describe a particular outcome with 3 heads, such as "HHTTHT" One needs to choose 3 of the 6 coins to record as a head (H), and the remaining will be tails by virtue of not being heads. So the question is also asking "In how many ways can you choose 3 out of 6 coins to be a head?" which is $\binom{6}{3} = 20$.

How many outcomes have more heads than tails?

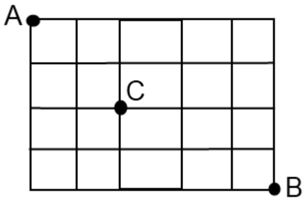
This type of problem asks you to count up a few distinct types of outcomes: how many outcomes have exactly 4 heads, how many have exactly 5 heads and how many have exactly 6 heads. From the previous example, these are calculated as $\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 15 + 6 + 1 = 22$.
Another way of thinking about this problem is as follows: We already know there are 64 possible coin flips, and 20 of those have exactly 3 heads. Therefore 44 of those flips do not have an equal number of heads and tails. It stands to reason that half of those would have more tails and half would have more heads, so the answer is $44/2 = 22$.

How many outcomes have at least 2 heads?

At least 2 heads means 2 or more heads. So we can count up: #flips with 2 heads + #flips with 3 heads + #flips with 4, 5 and 6 heads. Since we have already calculated The number of flips with 3 as 20, and the number of flips with 4 or more as 22, we just need the number of flips with 2 heads, which is

$\binom{6}{2} = 15$. So the total is $15 + 20 + 22 = 57$.

3) Based on the city map below, in how many ways can you take a direct (i.e. no back-tracking or going further away) route from A to B? (assume up is north)



No matter which way you go from A to B, you must go south 4 times and east 5 times. Therefore, a set of directions describing one particular route would read like "South, east, south, east, east, south, east, east, south." for example - let's use S and E for simplicity. Just like the coin toss problem, you have 9 steps of the route which can either be a S or an E (like H / T). Describing one route is akin to choosing 4 of these steps to be "S" which makes the remaining 5 to be E. For example, my route looks like this without any directions yet:

_ _ _ _ _ _ _ _ _

After choosing 4 blanks to be "S" we have:

S _ _ S _ S _ S _

And the remaining 5 have to be E, so we have: S E E S E S E S E as the steps in the route. How many ways can we choose 4 of 9 steps to be S? That's exactly $\binom{9}{4} = 126$. What if instead of choosing the S steps we choose the E and let the remaining 4 be south? That would be $\binom{9}{5} = 126$ also - the answer will be the same.

What if you need to stop at point C on the way? How many routes will pass through point C?

Always see if you can think of these questions as a sequence of decisions. In this case we can break it down into 2 parts - how many ways can you get from A to C, and how many ways can you get from C to B. From A to C we restrict the route to the small area of the map with corners A and C - it's a 2x2 grid. A route through this part of the city must have 4 steps and 2 of them must be "South". Thus there are $\binom{4}{2} = 6$ possible routes from A to C.

Likewise, from C to B you must travel through a 2x3 grid, a route of 5 steps 2 of which must be "South". This gives $\binom{5}{2} = 10$ routes from B to C. By the multiplication principal, the final answer is:

(# Routes from A to C)$\cdot$(# Routes from C to B) $= 6 \cdot 10 = 60$.

4) In how many ways can a committee of 5 senators be formed from the US Senate if the 5 senators must each represent a different state?

The US Senate has 100 senators, 2 from each state. This problem thus requires you to make a series of decisions, namely:

a) Which 5 states will the senators represent, and

b) Once the 5 states have been picked, which of the 2 senators from each state will be on the committee?

The first number is calculated by asking "In how many ways can 5 states be chosen from 50?" Which is $\binom{50}{5}$. The second number is calculated from a series of decisions - Which senator will be chosen from State A, Which senator from State B, etc. There are 5 pairs of senators to make decisions about, and each one can be made in 2 ways. Thus there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$ ways of picking 1 senator from each of the 5 states.

The product is $\binom{50}{5} \cdot 2^5 = 67,800,320$.

5) From a deck of 52 cards, suits of $\heartsuit, \diamondsuit, \spadesuit$, and $\clubsuit$ with card values 2-10, J, Q, K and A, how many possible poker hands are there that are:

A Pair?

For "A Pair", two of the cards are the same value (e.g. $2\spadesuit, 2\diamondsuit$) while the other three cards have different values and any suits (e.g. $7\heartsuit, K\heartsuit, 3\clubsuit$). We want to use the multiplication principal to break this down into a sequence of decisions that will count all possibilities:
a) What value will the pair be?
b) What 2 suits will the pair be?
c) What 3 distinct values will the other 3 cards be?
d) What will their suits be?
There are 13 possible values for the pair, and we choose 1, so the first number is $\binom{13}{1} = 13$.
To choose 2 suits from the 4 possible there are $\binom{4}{2}$ possibilities.
We must pick 3 of the 12 unused card values for the other 3 cards, which is $\binom{12}{3}$.
Lastly, because the three cards can have any suit, and there are 4 suits, this is found using the multiplication principal - $4 \cdot 4 \cdot 4 = 4^3$.
So the answer is $13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,099,240$.

Two Pairs?

The decisions for this problem are:
a) What 2 values will we have as pairs?
b) what 2 suits will be in the first pair?
c) What 2 suits will be in the second pair?
d) what will the last card be?
To choose 2 values from 13 for our two pairs, there are $\binom{13}{2}$ ways.
To choose 2 of 4 suits for the first pair, there are $\binom{4}{2}$ ways, and $\binom{4}{2}$ ways for the second pair.
Finally there are 11 unused values in 4 suits = 44 cards that are possible for the fifth card in the hand.
So the product is : $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123,552$ possible ways to be dealt 2 pair.

3 of a Kind?

The decisions for this problem are:
a) What value will the triple be?
b) What 3 suits will the triple have?
c) What values will the other 2 cards have?
d) what suits will the other 2 cards have?
Again, there are $\binom{13}{1} = 13$ possible values for the triple.
To choose 3 suits from the 4 available for the triple, we have $\binom{4}{3}$ possible ways.
There are $\binom{12}{2}$ possible values for the other 2 cards, and as in the Pair example there are $4 \cdot 4 = 4^2$ possible ways to assign them suits.
The answer is the product: $13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2 = 54,912$.

Royal Flush (10, J, Q, K, A of a single suit)?

The only decision for this problem is: What is the suit of the royal flush?
There are only 4 suits, so there are only 4 possible royal flushes that can be dealt!

Straight Flush (5 in a row of the same suit. Ace can be low or high)?

The decisions to make are:
a) What is the first card in the straight (the other 4 cards must be the next 4 in sequence)?
b) What suit is the flush?
If we listed all card values in ascending order, with A as both low and high card, we get A,2,3,4,5,6,7,8,9,10,J,Q,K,A.
You can check that a straight could possibly begin with A through 10, but it could not begin with a J because there are only 3 cards after J. There are 10 possibla cards for the straight to begin with.

There are 4 possible suits for the flush. Therefore there are $10 \cdot 4 = 40$ possible straight flushes (note: this includes the 4 royal flushes we have already counted - a royal flush is a special kind of straight flush).

A Straight (5 cards in a row, and not counting Straight Flushes)?
The decisions to make are:
a) What card will be the low card of the straight (the other 4 have to be the next 4 in sequence)?
b) What 5 suits will the cards have?
Like in the straight flush example, there are 10 possible cards for the straight to start with. Furthermore, we can allow for any of the 4 suits for each of the cards. This is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$. The product of these is $10 \cdot 4^5$, but wait - we have to subtract the number of straight flushes from this total. Thus, there are $10 \cdot 4^5 - 40 = 10,200$ possible straights, not counting straight flushes.

A Flush (5 cards all of the same suit, and not counting Straight Flushes)?
The decisions to make are:
a) What suit will the flush be?
b) What 5 card values we have?
There are 4 possible suits, and because all 5 cards are of the same suit it is impossible to repeat a card value - therefore we have to choose 5 different values from the 13 available. This is $\binom{13}{5}$. The product is $4 \cdot \binom{13}{5}$. But we have to remember to not count the Straight Flushes (there are 40 of them). So we subtract 40 from our total. Therefore, we can count $4 \cdot \binom{13}{5} - 40 = 5,108$ possible Flushes.