Math 160 , Finite Mathematics for Business

Section 5.8 - Discussion Notes
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Review Question

1) If you have 15 sweaters to pack away into a large box that can hold 10 and a small that can hold 5, in how many ways can you pack the sweaters away?

This is solved by choosing 10 sweaters from the 15 to put into the larger box. $\binom{15}{10} = \frac{15!}{10!5!} = 3,003$.

This type of problem is actually a special case of an **ordered partitioning** problem, where a set S of n elements is partitioned into ordered (i.e. distinct), non-overlapping subsets $S_1, S_2, ..., S_m$ of $n_1, n_2, ..., n_m$ elements respectively, and $n_1 + n_2 + ... + n_m = n$. In the above example there are only 2 subsets, one of 10 elements and the other of 5.

The general formula for making an **Ordered Partition** of a n elements into groups of $n_1, n_2, ..., n_m$ (which is called an ordered partition of type $(n_1, n_2, ..., n_m)$), with $n_1 + n_2 + ... + n_m = n$ is:

$$\binom{n}{n_1, n_2, ..., n_m} = \frac{n!}{n_1! n_2! \cdots n_m!}$$

A special case of this is when you partition the set into subsets of the same number of elements each. That is to say, if $n_1 = n_2 = ... = n_m = r = \frac{n}{m}$, the formula is:

$$\binom{n}{r, r, ..., r(mtimes)} = \frac{n!}{(r!)^m}$$

The notation $\binom{n}{n_1, n_2, ..., n_m}$ is known as a **multinomial coefficient**.

We also consider problems where the partitions are **unordered** - this means there is no distinguishing between one partition and another. In this course we only consder examples where n things are partitioned into m unordered groups of the same r things in each. The formula is:

$$\frac{1}{m!} \frac{n!}{(r!)^m}$$

2) In how many ways can you partition a set S of 5 elements into an ordered partition of type (2, 2, 1)?

This is a direct application of the formula.
$\binom{5}{2,2,1} = \frac{5!}{2!2!1!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)(1)} = 30$.

3) Example revisited

In how many ways can you assign 100 students to dorms A, B and C, with 25 to dorm A, 40 to B and 35 to C?

This is an example of an ordered partition problem. We can write the multinomial coefficient as $\binom{100}{25,40,35} = \frac{100!}{25!40!35!} \approx 7.14 \times 10^{44}$.

4) If S is a set of 7 elements, how many ordered partitions are there of type (3, 2, 2)?

Plugging this into the formula, we get $\binom{7}{3,2,2} = \frac{7!}{3!2!2!} = 210$.

5) If S is a set of 10 elements, how many unordered partitions are there of type (5, 5)?

This problem is asking about unordered partitions. In this example, n=10 (10 elements in the main set), m=2 (there are 2 partitions) and r=5 (there are 5 elements going into each partition). The formula is $\frac{1}{m!} \cdot \frac{n!}{(r!)^m} = \frac{1}{2!} \cdot \frac{10!}{(5!)^2} = 126$.

6) In how many ways can an investment rating company rate 15 stocks A, AA and AAA if it wants to put 5 into each category?

Because the three partitions are distinctly labeled (A, AA and AAA) this is an ordered partition of type (5, 5, 5). This can be done in:
$\binom{15}{5,5,5} = \frac{15!}{5!5!5!} = 756,756$ ways.

7) In how many ways can 4 employees be promoted if one will become president, 1 will become vice president and 2 will be placed on the board of directors?

The partitions are distinctly labeled, so this is an ordered partition of type (1, 1, 2). This is calculated by $\binom{4}{1,1,2} = \frac{4!}{1 \cdot 1 \cdot 2!} = 12$. So it can be done in 12 ways.

8) 14 children are going on a field trip and must be paired up (the buddy system). In how many ways can the 14 kids be paired up with each other?

This example is an unordered partition, because there is no labeling of the groups - we are not calling one pair "Pair 1" and another pair "Pair 2" - If Sally and Joe are paired together, that's all that matters. So we must use the unordered partition formula. We are partitioning 14 children into 7 groups of 2. So n=14, m=7 and r=2. The formula gives us the count of $\frac{1}{7!}\frac{14!}{(2!)^7} = 135,135$ ways of pairing up the kids.

9) 10 members of city council decide to form 2 committees of 6 people each (2 city council members must serve on both committees). The first committee is the Zoning Committee and the second is the Street Repairs Committee. In how many ways can the city council members break up into these two committees?

We do not have a formula for overlapping partitions, so we have to think about this in a way such that we have non-overlapping partitions. Therefore consider the members dividing into 3 paritions:
a) 4 members who serve only on the Zoning Committee,
b) 4 members who serve only on the Street Repairs Committee, and
c) 2 members who serve on both committees.
These 3 partitions are definitely non-overlapping and these numbers fit neatly into our formula - we've made this problem into a partition of type (4, 4, 2) which is calculated as: $\binom{10}{4,2,2} = \frac{10!}{4!4!2!} = 3,150$.