Math 180: Calculus I **Fall 2014**

October 14

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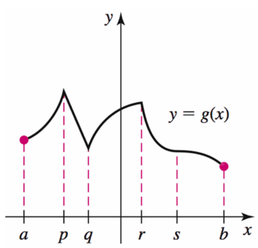
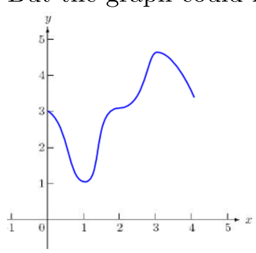
1. On the following graph to determine at what x values on the interval $[a, b]$ local and absolute extreme values occur.



SOLUTION: Local minima at $x = q$. Local maxima at $x = p, r$. Absolute minimum at $x = b$. Absolute maximum at $x = p$.

2. Sketch the graph of a function on the interval $[0, 4]$ with the following properties:
$f'(x) = 0$ for $x = 1, 2,$ and 3; f has an absolute minimum at $x = 1$; f has no local extremum at $x = 2$; and f has an absolute maximum at $x = 3$.
SOLUTION: Of course there are any number of correct graphs, it doesn't even need to be continuous. But the graph could look like this.



It's important that $f(1)$ is the least value the function takes on the interval $[0, 4]$ and $f(3)$ is the greatest value it takes on the interval. The curve must level off at $x = 2$ momentarily.

3. Find the critical points of the following functions on the domain given, and try to classify each as a local minimum, maximum or neither.

(a) $f(x) = 3x^2 - 4x + 2$ on $(-\infty, \infty)$
SOLUTION: $f'(x) = 6x - 4$, so the only critical point is $x = 2/3$, where the derivative is zero.

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2 = \frac{2}{3}$$

Because this curve is a parabola which opens up, $(\frac{2}{3}, \frac{2}{3})$ must be a local minimum.

(b) $f(x) = (e^x + e^{-x})/2$ on $(-\infty, \infty)$
SOLUTION: $f'(x) = (e^x - e^{-x})/2$. Setting this equal to zero we get $x = 0$ is the only critical value of x.

$$f(0) = \frac{1+1}{2} = 1$$

Because the function tends to infinity both as $x \to \infty$ and $x \to -\infty$, $(0, 1)$ is a local minimum.

(c) $f(x) = \sin x \cos x$ on $[0, 2\pi]$
$f'(x) = \sin x(-\sin x) + \cos x \cos x$. Setting this equal to zero we get

$$\cos^2 x = \sin^2 x,$$

or

$$|\cos x| = |\sin x|.$$

This is true when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The endpoints, $x = 0, 2\pi$ must be considered as well.

Point	Classification	Reason
$(0,0)$	neither	not a minimum or maximum value
$(\frac{\pi}{4}, \frac{1}{2})$	local max	continuous function, derivative zero
$(\frac{3\pi}{4}, -\frac{1}{2})$	local min	"
$(\frac{5\pi}{4}, \frac{1}{2})$	local max	"
$(\frac{7\pi}{4}, -\frac{1}{2})$	local min	"
$(2\pi, 0)$	neither	not a minimum or maximum value

4. Find the critical points of f on the given interval and determine the absolute extreme values of f if they exist.

(a) $f(x) = x(x^2+1)^{-2}$ on $[-2, 2]$
SOLUTION: $f'(x) = (x^2+1)^{-2} + -2x(x^2+1)^{-3}(2x)$. Setting this equal to zero we get

$$0 = \frac{1}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^3}$$

Multiplying by $(x^2+1)^3$ we get

$$0 = x^2 + 1 - 4x^2$$

Which is solved when $x = \pm\sqrt{1/3}$. We have to check these two x values as well as the endpoints.

$$f(-2) = -2/5 = -.4, f(-\sqrt{1/3}) = -\frac{9}{16\sqrt{3}} \approx -.325, f(\sqrt{1/3}) \approx .325, f(2) = .4$$

So the absolute extreme values of f are $-.4$ and $.4$, which occur at the endpoints.

(b) $f(x) = \sin(3x)$ on $[-\pi/4, \pi/3]$
SOLUTION: $f'(x) = 3\cos(3x)$, which is zero when $x = ..., -3\pi/6, -\pi/6, \pi/6, 3\pi/6, ...$ but the only values on the given interval are $x = -\pi/6, \pi/6$. Checking these and th endpoints we have

$$f(-\pi/4) = -\sqrt{2}/2, f(-\pi/6) = -1, f(\pi/6) = 1, f(\pi/3) = 0$$

So 1 and -1 are the absolute extreme values of f.

(c) $f(x) = x\ln(x/5)$ on $[0.1, 5]$
SOLUTION: $f'(x) = \ln(x/5) + x\frac{1}{x/5}(1/5) = \ln(x/5) + 1$. Solving for x we get $x = 5e^{-1} \approx 1.84$, which is within the interval in question. We check this and the two endpoints:

$$f(0.1) \approx -.621, f(5e^{-1}) = -5e^{-1} \approx -1.84, f(5) = 0$$

So the absolute extreme values of f on the given interval are $-5e^{-1}$ and 0.

5. Find the local and extreme values of $f(x) = |x-3| + |x+2|$ on $[-4, 4]$.
SOLUTION: This can be written as a piecewise function, with breaks at $x = 3$ and $x = -2$.

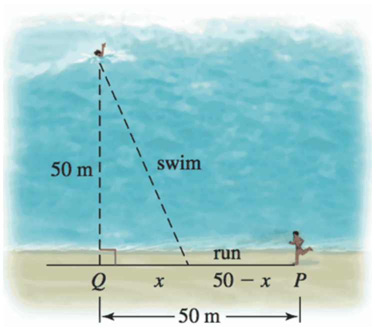
$$f(x) = \begin{cases} x - 3 + x + 2 & \text{if } x \geq 3 \\ 3 - x + x + 2 & \text{if } -2 \leq x < 3 \\ 3 - x - 2 - x & \text{if } x < 2 \end{cases}$$

Which can be simplified to

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \geq 3 \\ 5 & \text{if } -2 \leq x < 3 \\ 1 - 2x & \text{if } x < -2 \end{cases}$$

At the endpoints we have $f(-4) = 9, f(4) = 7$. So 9 is an absolute maximum of the function, 5 is a local and absolute minimum, and that is it.

6. You are running along the shore from point P towards point Q which is 50m away. 50m from Q perpendicular to the shore, there is a drowning swimmer. You can run at 4m/s and swim at $2m/s$. At what point x meters from Q should you jump into the water to swim if you want to minimize the time to get to the swimmer?



SOLUTION: First of all, the distance that you swim is found by the Pythagorean Theorem, $\sqrt{x^2 + 50^2} = \sqrt{x^2 + 2500}$. To get seconds/meter, we take the reciprocals of th speeds. Thus the total time taken to get to the swimmer is

$$f(x) = \frac{1}{4}(50 - x) + \frac{1}{2}\sqrt{x^2 + 2500} = 10.25 - \frac{x}{4} + \frac{(x^2 + 2500)^{1/2}}{2}$$

We take the derivative to find a minimum:

$$f'(x) = -\frac{1}{4} + \frac{1}{2}\frac{(x^2 + 2500)^{-1/2}}{2}(2x) = -\frac{1}{4} + \frac{x}{2\sqrt{x^2 + 2500}}$$

Setting this equal to zero we get

$$\frac{1}{4} = \frac{x}{2\sqrt{x^2 + 2500}}$$

or

$$\sqrt{x^2 + 2500} = 2x$$

We square both sides and get

$$(x^2 + 2500) = 4x^2$$

$$3x^2 = 2500$$

We get $x \approx 28.868$. Checking the function value at this and the endpoints $x = 0$ and $x = 50$ we get

$$f(0) \approx 35.25, f(28.868) \approx 31.9, f(50) = 35.56$$

So clearly you should swim when $x = 28.868$, or in other words after running for 21.132m (hopefully you can do all of this calculus while in the running).