Math 180: Calculus I **Fall 2014**

October 2

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1. Find $\frac{dy}{dx}$ using implicit differentiation

   (a) $\sin(xy) = x + y$
   SOLUTION:

$$\begin{aligned} \cos(xy)(y + xy') &= 1 + y' && \text{take derivative of both sides} \\ \cos(xy)y + \cos(xy)xy' &= 1 + y' && \text{distribute the left side} \\ y'x\cos(xy) - y' &= 1 - y\cos(xy) && \text{collect all terms with } y' \text{ as a factor} \\ y'(x\cos(xy) - 1) &= 1 - y\cos(xy) && \text{factor out } y' \\ y' &= \frac{1 - y\cos(xy)}{x\cos(xy) - 1} && \text{solve for } y' \end{aligned}$$

   (b) $\cos(y^2) + x = e^y$
   SOLUTION:

$$\begin{aligned} -\sin(y^2)2yy' + 1 &= e^y(y') && \text{take derivative of both sides, use chain rule} \\ y'(-2y\sin(y^2)) - y'e^y &= -1 && \text{collect terms with } y' \text{ as a factor} \\ y'(-2y\sin(y^2) - e^y) &= -1 && \text{factor out } y' \\ y' &= \frac{-1}{-2y\sin(y^2) - e^y} && \text{solve for } y' \\ y' &= \frac{1}{2y\sin(y^2) + e^y} && \text{simplify} \end{aligned}$$

   (c) $y = \frac{x+1}{y-1}$
   SOLUTION:

$$\begin{aligned} y' &= \frac{(1)(y-1) - (x+1)(y')}{(y-1)^2} && \text{take derivative of both sides, quotient rule} \\ y' &= \frac{y-1}{(y-1)^2} - y'\frac{x+1}{(y-1)^2} && \text{split the fraction} \\ y' + y'\frac{x+1}{(y-1)^2} &= \frac{y-1}{(y-1)^2} && \text{collect } y' \text{ terms} \\ y'\left(1 + \frac{x+1}{(y-1)^2}\right) &= \frac{1}{y-1} && \text{factor the } y' \text{ and simplify RHS} \\ y' &= \frac{1}{y-1}\left(1 + \frac{x+1}{(y-1)^2}\right)^{-1} && \text{solve for } y' \end{aligned}$$

   This can be simplified, but that is not necessary.

2. Find the slope at the given point.

(a) $\sqrt[3]{x} + \sqrt[3]{y^4} = 2; (1,1)$
SOLUTION: First we must find the derivative

$$\begin{aligned} x^{1/3} + y^{4/3} &= 2 && \text{rewrite as exponents} \\ \frac{1}{3}x^{-2/3} + \frac{4}{3}y^{1/3}y' &= 0 && \text{take derivative of both sides} \\ y'\frac{4y^{1/3}}{3} &= -\frac{1}{3}x^{-2/3} && y' \text{ term alone} \\ y' &= -\frac{x^{-2/3}}{4y^{1/3}} && \text{solve for } y' \end{aligned}$$

We plug in the point $(1,1)$:

$$y'|_{(1,1)} = -\frac{(1)^{-2/3}}{4(1)^{1/3}} = -\frac{1}{4}$$

(b) $(x+y)^{2/3} = y, (4,4)$

$$\begin{aligned} \frac{2}{3}(x+y)^{-1/3}(1+y') &= y' && \text{take derivative of both sides} \\ \frac{2}{3}(4+4)^{-1/3}(1+y') &= y' && \text{Plug in the point (4,4)} \\ \frac{1}{3}(1+y') &= y' && \text{clean up the fraction} \\ \frac{1}{3} &= y' - \frac{1}{3}y' && y' \text{ terms to one side} \\ \frac{1}{3} &= \frac{2}{3}y' && \text{subtract on Right} \\ \frac{1}{2} &= y' \end{aligned}$$

3. Find the equations of each tangent line for $x = 1$ for the following curve

$$x + y^3 - y = 1$$

SOLUTION: First we need to determine what possible y values there are for $x = 1$.

$$\begin{aligned} (1) + y^3 - y &= 1 \\ y(y^2 - 1) &= 0 \\ y(y-1)(y+1) &= 0 \end{aligned}$$

So the points are $(1,0), (1,-1)$, and $(1,1)$. Now we use implicit differentiation to find y'.

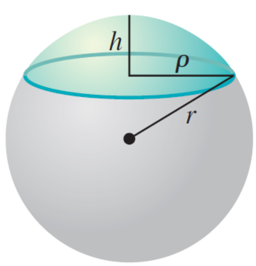
$$\begin{aligned} 1 + 3y^2y' - y' &= 0 && \text{derivative} \\ y'(3y^2 - 1) &= -1 && \text{collect terms and factor} \\ y' &= \frac{-1}{3y^2 - 1} \end{aligned}$$

Plug in the y values and we get the slopes $y' = 1, -1/2$, and $-1/2$ respectively. So the three tangent lines are given by

$$y = 1(x-1), \quad y + 1 = -\frac{1}{2}(x-1), \text{and} \quad y - 1 = -\frac{1}{2}(x-1)$$

4. (a) At what point does $x + y^3 - y = 1$ have a vertical tangent line? (b) Does it have any horizontal tangent lines?
SOLUTION: A vertical tangent line occurs when y' is undefined because the curve is continuous. From the derivative this is clearly when $3y^2 - 1 = 0$ or when $y^2 = 1/3$, i.e. $y = \pm\sqrt{1/3}$. We must find the corresponding x values. We may solve for x with
$$x = 1 + y - y^3$$
So the corresponding x values are
$$1 + \sqrt{\frac{1}{3}} - \sqrt[3]{\frac{1}{3}} \quad \text{and} \quad 1 - \sqrt{\frac{1}{3}} + \sqrt[3]{\frac{1}{3}}$$
Thus the points where the tangent line is vertical are
$$\left(1 + \sqrt{\frac{1}{3}} - \sqrt[3]{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right) \quad \text{and} \quad \left(1 - \sqrt{\frac{1}{3}} + \sqrt[3]{\frac{1}{3}}, -\sqrt{\frac{1}{3}}\right)$$
The derivative y' cannot be made zero, though, no matter what value of y is chosen, so there are no points where we have a horizontal tangent line.

5. If you slice a sphere the small piece is a spherical cap. Its volume is given by
$$V = \frac{1}{3}\pi h^2(3r - h)$$
where r is the radius of the sphere and h is the cap thickness.
   (a) Find $\frac{dr}{dh}$ for a spherical cap of volume $\frac{5\pi}{3}$.
   (b) Evaluate the derivative $\frac{dr}{dh}$ when $r = 2$ and $h = 1$.



SOLUTION: If we hold the volume of the cap fixed at $5\pi/3$, then the equation is
$$\frac{5\pi}{3} = \frac{1}{3}\pi h^2(3r - h)$$
Which simplifies to
$$5 = 3rh^2 - h^3$$
We take the derivative of both sides to solve for $r' = \frac{dr}{dh}$
$$0 = 3r'h^2 + 6rh - 3h^2$$
We solve for r' and get
$$r' = \frac{3h^2 - 6rh}{3h^2} = \frac{h - 2r}{h}$$
For part (b), we plug in $r = 2, h = 1$ and get
$$r' = \frac{(1) - 2(2)}{(1)} = -3$$