Math 180: Calculus I **Fall 2014**

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TA: Brian Powers

When graphing functions, there is some basic analysis that will help you do it.

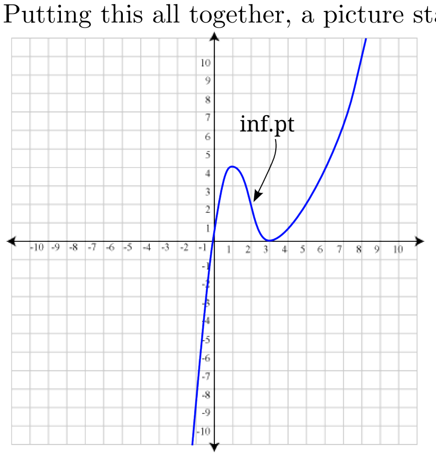
- Identify the domain or the interval in question
- identify if there is any helpful symmetry (e.g. even/odd function)
- Find critical points & inflection points
- Find the extreme values of the function
- Identify any asymptotes, or the end behavir as $x \to \infty$ or $-\infty$.
- Find the x and y intercepts

1. Graph the following functions

   (a) $f(x) = x^3 - 6x^2 + 9x$
   SOLUTION: First there is no symmetry since it is not an even function or an odd function. The domain is $(-\infty, \infty)$. The y-intercept is $f(0) = 0$, and the x intercepts are found by setting

$$0 = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)(x-3)$$

So it has x-intercepts at $x = 0, 3$.
The first derivative is $f'(x) = 3x^2 - 12x + 9$, setting this equal to zero we get $0 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$, so we have critical points at $x = 1, 3$. Furthermore, a test of the sign of $f'(x)$ on the intervals gives us that f is increasing on $(-\infty, 1), (3, \infty)$ and decreasing on $(1, 3)$. The values of f at its critical points are $f(1) = 1 - 6 + 9 = 4$, $f(3) = 0$, as we already know. These are a local minimum and maximum respectively.
The second derivative is $f''(x) = 6x - 12$ which has a single zero at $x = 2$, which is an inflection point separating the interval of downward concavity $(-\infty, 2)$ from upward concavity $(2, \infty)$. Putting this all together, a picture starts to emerge.



(b) $f(x) = \frac{4x+4}{x^2+3}$

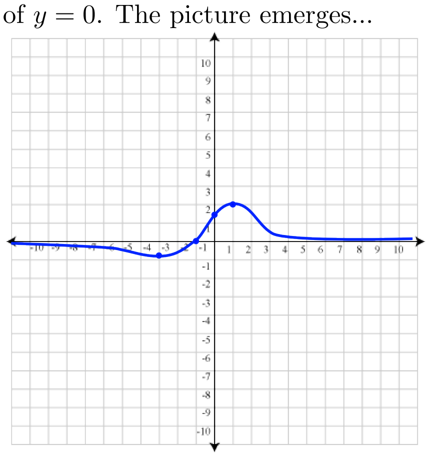
SOLUTION: No symmetry to help us with this graph. We can find intercepts though: $f(0) = 4/3$ and if we set this equal to zero, we get $0 = 4x + 4$ so an x-intercept at $x = -1$. Since the denominator cannot ever be zero, we have no vertical asymptotes, and the domain is all real numbers.

Taking a first derivative, we get

$$f'(x) = \frac{4(x^2+3) - (4x+4)(2x)}{(x^2+3)^2} = \frac{4x^2 + 12 - 8x^2 - 8x}{(x^2+3)^2} = \frac{-4(x^2+2x-3)}{(x^2+3)^2}$$

We get $f'(x) = 0$ at $x = -3, 1$, which we get from the numerator. Since the denominator is always positive, we can get the intervals of increasing/decreasing from the numerator of f' as well. Since it is a parabola opening downwards, it will have negative sign from $(-\infty, -3)$ and $(1, \infty)$ with positive sign on the interval $(-3, 1)$. These are the intervals of decreasing and increasing respectively. So we have $x = -3$ is a local minimum, $x = 1$ is a local maximum. Plugging these into f we get $f(-3) = (-8/12) = -3/4$ and $f(1) = 8/4 = 2$.

Lastly, as the end behavior at negative and positive infinity is that the function goes to zero (because the denominator is a polynomial of higher degree). So we have a horizontal asymptote of $y = 0$. The picture emerges...



(c) $f(x) = 2 - x^{2/3} + x^{4/3}$

SOLUTION: This function at least is an even function, so we have symmetry. $f(0) = 2$, and are there any zeroes? Let $u = x^{2/3}$, $0 = 2 - u + u^2$. The quadratic formula gives

$$u = \frac{1 \pm \sqrt{1-8}}{4}$$

which has no real values. So this function does not have any zeroes, there are no x intercepts.

$$f'(x) = -\frac{2}{3}x^{-1/3} + \frac{4}{3}x^{1/3}$$

When we set this equal to zero, we have

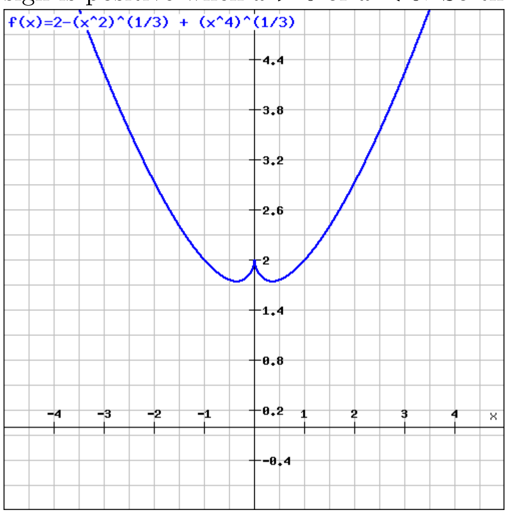
$$\frac{2}{3}x^{-1/3} = \frac{4}{3}x^{1/3}$$
$$x^{-1} = 8x$$

So $x = \frac{1}{\sqrt{8}}$ or $-\frac{1}{\sqrt{8}}$. Also $x = 0$ is a critical point since the first derivative does not exist there. The first derivative goes to ∞ as $x \to \infty$ or $x \to -\infty$ since the first term goes to zero.

$f(8^{-1/2}) = 2 - 8^{-1/3} + 8^{-2/3} = 2 - \frac{1}{2} + \frac{1}{4} = 1.75 = f(-8^{-1/3})$ (by symmetry). So we have $x = \pm 8^{-1/3}$ are local minima, $x = 0$ is a local max, but also a cusp.

$$f''(x) = \frac{2}{9}x^{-4/3} + \frac{4}{9}x^{-2/3}$$

The second derivative is undefined at $x = 0$ but because both terms are even powers of x, the sign is positive when $x > 0$ or $x < 0$. So the intervals of upward concavity are $(-\infty, 0), (0, \infty)$.



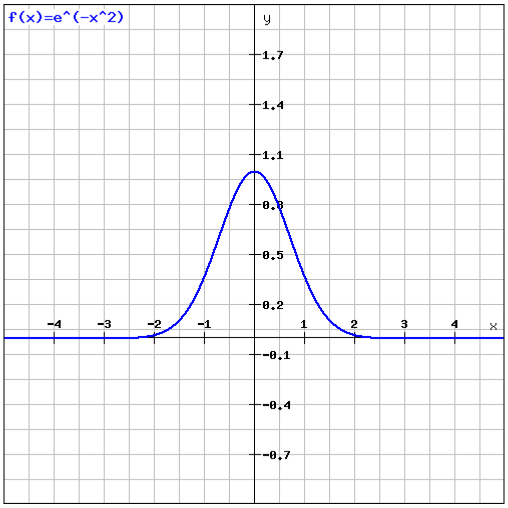
(d) $s(x) = e^{-x^2}$
SOLUTION: This function is an even function, so we have horizontal symmetry. The exponential function has no zeroes, but $f(0) = e^0 = 1$. Also we have a horizontal asymptote of $y = 0$, since as $x \to \infty$, $f(x) \to 0$.

$$f'(x) = e^{-x^2}(-2x)$$

and the only critical point is $x = 0$, which as we know is a local maximum (since $f(0) = 1$ and the function goes to zero to the left and to the right).

$$f''(x) = e^{-x^2}(-2) + e^{-x^2}(-2x)(-2x) = (4x^2 - 2)e^{-x^2}$$

So we have $f''(x) = 0$ when $x = \pm\sqrt{1/2}$ Since $x = 0$ is a local maximum, we know that the function is concave down on the interval $(-\sqrt{1/2}, \sqrt{1/2})$. By testing $x = 2$ we have $f''(2) > 0$ so the function is concave up on $(-\infty, -\sqrt{1/2}), (\sqrt{1/2}, \infty)$. This is enough to give us the picture.



2. Find a possible graph for $f(x)$ based on its derivative:

$$f'(x) = (x-1)(x+2)(x+4)$$

From the first derivative we know we have critical points at $x = 1, -2, -4$ and by a sign graph we have the intervals of increasing are $(-4,-2), (1,\infty)$ and decreasing on $(-\infty,-4), (-2,1)$.

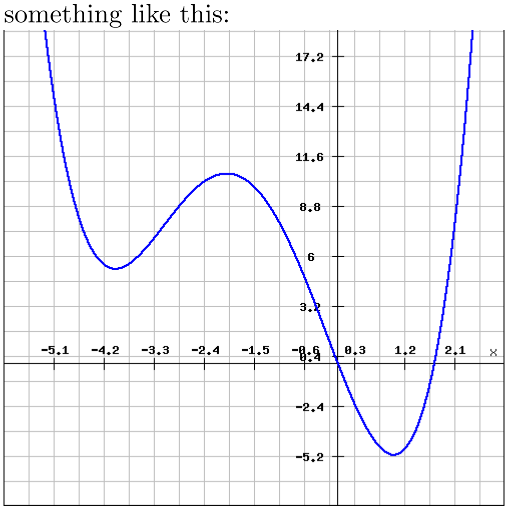
To find the second derivative, let's multiply out the first derivative.

$$\begin{aligned} f'(x) &= (x^2+x-2)(x+4) \\ &= (x^3+x^2-2x)+(4x^2+4x-8) \\ &= x^3+5x^2+2x-8 \end{aligned}$$

So $f''(x) = 3x^2+10x+2$. Setting this equal to zero we have inflection points at

$$x = \frac{-10 \pm \sqrt{100-(4)(3)(2)}}{6} = \frac{-10 \pm \sqrt{76}}{6}$$

Since $\sqrt{76} \approx 9$, we have inflection points near $x = -19/6$ and $x = -1/6$. This should look like a W, something like this:



3. Sketch a graph of $f(x) = x^3 - 3x^2 - 144x - 140$. What property makes it "easy" to graph?
SOLUTION: The textbook thinks this has a property that makes it "easy" to graph. The only thought I have is that it has an integer root which you can find by testing 0, $\pm1, \pm2$. In fact, if you plug in $x = -1$ you get 0, so you are able to factor out $(x + 1)$. This is the foothold you need in order to get some other zeroes.

By synthetic division or polynomial division, we get

$$f(x) = (x + 1)(x^2 - 4x - 140) = (x + 1)(x - 14)(x + 10)$$

So I suppose having integer roots is a nice property. We also have $f(0) = -140$. Since all roots are of multiplicity 1, we know that the graph crosses the x axis at $x = -10, -1, 14$. Let's find the local extrema:

$$f'(x) = 3x^2 - 6x - 144$$

set equal to zero gives 0=3(x-8)(x+6), we have critical points at $x = -6, 8$ which must be local maximum and minimum respectively. Why?

The curve starts negative (plug in $x = -100$ for example). It is increasing, crosses the x axis at $x = -10$, then must stop increasing and decrease to cross the x axis at $x = 1$. Then it must stop decreasing and increase, crossing again at $x = 14$.

The graph will look like this: