Math 180: Calculus I **Fall 2014**

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We may use the following derivative rules now:

$$\frac{d}{dx}b^x = b^x \ln(b) \quad \frac{d}{dx}\ln x = \frac{1}{x} \quad \frac{d}{dx}\ln|u(x)| = \frac{u'(x)}{u(x)} \quad \frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

And the technique of logarithmic differentiation: take a log of both sides of the equation, then take the derivative using implicit differentaition to solve for $f'(x)$.

1. Find the following derivatives

   (a) $\frac{d}{dx}(x^2 \ln x)$
   SOLUTION:
   $$= 2x \ln x + x^2 \frac{1}{x}$$

   (b) $\frac{d}{dx}x^3 3^x$
   SOLUTION:
   $$= 3x^2 3^x + x^3(3^x \ln 3)$$

   (c) $\frac{d}{dx}(\ln|\sin x|)$
   SOLUTION:
   $$= \frac{\cos x}{\sin x} = \cot x$$

   (d) $\frac{d}{dx}\ln(10^x)$
   SOLUTION:
   $$= \frac{10^x \ln 10}{10^x} = \ln 10$$
   Or you may use hte fact that
   $$\ln(10^x) = x \ln 10$$
   and simply take that derivative ($\ln 10$ is just a constant).

   (e) $\frac{d}{dx}(\ln(\ln x))$
   SOLUTION:
   $$= \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

2. Find the derivatives

   (a) $s(t) = \cos(2^t)$
   SOLUTION:
   $$s'(t) = -\sin(2^t)(2^t \ln t)$$

   (b) $f(x) = \ln\left[(x^3+1)^\pi\right]$
   SOLUTION:
   $$f'(x) = \frac{\pi(x^3+1)^{\pi-1}(3x^2)}{(x^3+1)^\pi}$$

3. Evaluate the derivative of $h(x) = x^{\sqrt{x}}$ at $x = 4$.
SOLUTION: Using logarithmic differentiation, we take the natural log of both sides first

$$\begin{aligned}
\ln h(x) &= \ln\left(x^{\sqrt{x}}\right) \\
\ln h(x) &= \sqrt{x}\ln x && \text{by log property} \\
\frac{h'(x)}{h(x)} &= \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x} && \text{take derivative of both sides} \\
h'(x) &= h(x)\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right) && \text{solve for } h'(x) \\
h'(x) &= x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right) && \text{substitute in } h(x) \\
h'(4) &= 4^{\sqrt{4}}\left(\frac{\ln 4}{2\sqrt{4}} + \frac{1}{\sqrt{4}}\right) && \text{plug in 4} \\
&= 16\left(\frac{\ln 4}{4} + \frac{1}{2}\right) \\
&= 4\ln 4 + 8
\end{aligned}$$

4. Find the horizontal tangent line equation for $y = x^{\ln x}$
SOLUTION: Using logarithmic differentiation,

$$\begin{aligned}
\ln y &= \ln\left(x^{\ln x}\right) \\
\ln y &= (\ln x)(\ln x) = (\ln x)^2 && \text{by log property} \\
\frac{y'}{y} &= 2(\ln x)\frac{1}{x} && \text{take derivative of both sides} \\
y' &= y\frac{2\ln x}{x} && \text{solve for } y' \\
y' &= x^{\ln x}\frac{2\ln x}{x} && \text{substitute in } y
\end{aligned}$$

Because $x = 0$ is not in the domain, the only way this derivative can be zero is for $\ln x = 0$. This happens when $x = 1$ (this is true for all logs, no matter what base). Plugging in 1 for x we get

$$y = (1)^{\ln 1} = 1^0 = 1$$

So the equation of the horizontal tangent line is

$$y = 1$$

5. Use logarithmic differentiation to find the derivative of

$$f(x) = \frac{x^8\cos^3 x}{\sqrt{x-1}}$$

SOLUTION:

$$\begin{aligned}
\ln f(x) &= \ln\left(\frac{x^8\cos^3 x}{\sqrt{x-1}}\right) && \text{take log of both sides}\\
\ln f(x) &= \ln(x^8) + \ln(\cos^3 x) - \ln(\sqrt{x-1}) && \text{by log properties}\\
\ln f(x) &= 8\ln x + 3\ln(\cos x) - \frac{1}{2}\ln(x-1) && \text{by log properties}\\
\frac{f'(x)}{f(x)} &= \frac{8}{x} - 3\frac{\sin x}{\cos x} - \frac{1}{2}\frac{1}{x-1} && \text{take derivative of both sides}\\
&= \frac{8}{x} - 3\tan x + \frac{1}{2-2x}\\
f'(x) &= f(x)\left(\frac{8}{x} - 3\tan x + \frac{1}{2-2x}\right) && \text{solve for } f'(x)\\
&= \left(\frac{x^8\cos^3 x}{\sqrt{x-1}}\right)\left(\frac{8}{x} - 3\tan x + \frac{1}{2-2x}\right) && \text{substitute back } f(x)
\end{aligned}$$

6. Find the derivative y' of

$$y = (x^2+1)^x$$

using two methods:
(1) Use the fact that

$$b^x = e^{x\ln b}$$

(2) Use logarithmic differentiation.
SOLUTION:
Method (1):

$$y = e^{x\ln(x^2+1)}$$

So

$$y' = e^{x\ln(x^2+1)}\left(\ln(x^2+1) + x\frac{2x}{x^2+1}\right) = (x^2+1)^x\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$$

Method (2):

$$\begin{aligned}
\ln y &= \ln\left[(x^2+1)^x\right]\\
&= x\ln(x^2+1)\\
\frac{y'}{y} &= \ln(x^2+1) + \frac{x}{x^2+1}(2x)\\
y' &= y\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)\\
&= (x^2+1)^x\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)
\end{aligned}$$