Math 180: Calculus I **Fall 2014**

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TA: Brian Powers

1. Does Rolle's Theorem apply? If so find an x value on the interval with a horizontal tangent line.

   (a) $f(x) = x(x-1)^2 \quad [0,1]$
   SOLUTION: Since $f(0) = f(1) = 0$, and because the function is a polynomial (therefore differentiable everywhere, in particular on the interval given) Rolle's theorem applies. It is helpful to expand the function as $f(x) = x^3 - 2x^2 + x$, so the derivative is $f'(x) = 3x^2 - 4x + 1$. By the quadratic formula or by factoring you can find that $x = -1$ and $x = \frac{1}{3}$ are the two solutions. However, only $x = \frac{1}{3}$ is on the interval in question, so this is the point we are looking for.

   (b) $f(x) = \cos(4x) \quad [\pi/8, 3\pi/8]$
   SOLUTION: The cosine function is differentiable everywhere, and $f(\pi/8) = \cos(4\pi/8) = 0$, $f(3\pi/8) = \cos(12\pi/8) = 0$, so Rolle's theorem applies. $f'(x) = -4\sin(4x)$ and $-4\sin(4x) = 0$ as long as $4x = 0$ or $4x = \pi$. This gives us two solutions, $x = 0, \pi/4$ but only the latter is on the interval.

   (c) $f(x) = 1 - |x| \quad [-1,1]$
   SOLUTION: The function is not differentiable everywhere on the interval (specifically, at $x = 0$) so Rolle's Theorem cannot be used.

2. Does the Mean Value Theorem apply? If so find the point(s) guaranteed to exist.

   (a) $f(x) = 7 - x^2 \quad [-1,2]$
   SOLUTION: Polynomials are differentiable, so we can use the MVT. The average rate of change on the interval is
   $$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{(7 - 2^2) - (7 - (-1)^2)}{3} = -1$$
   and the derivative is $f'(x) = -2x$. Setting these equal we have $-2x = -1$ so the point guaranteed is $x = \frac{1}{2}$.

   (b) $f(x) = e^x \quad [0, \ln 4]$
   SOLUTION: The exponential function is differentiable everywhere, so there is no problem in using the MVT. The average rate of change on the interval is
   $$\frac{f(\ln 4) - f(0)}{\ln 4 - 0} = \frac{4 - 1}{\ln 4} = \frac{3}{\ln 4}$$
   If we set the derivative equal to the average rate of change, we get $e^x = \frac{3}{\ln 4}$ and taking a natural log of both sides gives us $x = \ln(\frac{3}{\ln 4})$, the point guaranteed by the MVT.

   (c) $f(x) = 3\sin(2x) \quad [0, \pi/4]$
   SOLUTION: The sine function is differentiable everywhere, so the MVT applies. The average rate of change is
   $$\frac{f(\pi/4) - f(0)}{\pi/4 - 0} = \frac{3}{\pi/4} = \frac{12}{\pi}$$
   Setting the derivative equal to the average rate of change we get $6\cos(2x) = 12/\pi$, which means $\cos(2x) = 2/\pi$. Since this is approximately $2/3$, we know that there is an x which solves it, so we can take an inverse cosine of both sides, giving $x = \cos^{-1}(2/\pi)$.

3. Which functions have the same derivative (without evaluating derivatives?)

$$f(x) = \ln x \quad g(x) = \ln 2x \quad h(x) = \ln(x^2) \quad p(x) = \ln(10x^2)$$

   SOLUTION: This question relies on the fact that two functions have the same derivative if they differ only by the addition of a constant. We can use properties of logs to rewrite $g(x) = \ln 2 + \ln x$ and $p(x) = \ln 10 + \ln x^2$. Then it is clear that $f(x)$ and $g(x)$ have the same derivative, $h(x)$ and $p(x)$ have the same derivative.

4. A car starts from rest at an intersection at an entrance ramp to a highway where the speed limit is 60 mph. At a checkpoint 30 miles away, 28 minutes later the car was clocked at 60 mph exactly.

   (a) How do we know the car was speeding?
   SOLUTION: The average rate of change (i.e. average speed) is 30mi/28min$\approx$ 64mph. Because the position of the car is a continuous, differentiable function of time, there must have been a point where the car was going exactly 64mph during the 30 miles.

   (b) What if it took 30 minutes for the car to reach the checkpoint. Can we still be sure the car was speeding?
   SOLUTION: For this we use the fact that the car started at 0 mph, and 30 minutes later had traveled 30 miles. The average speed was exactly 60 mph, but since the car had to accelerate from 0, it was going less than the average for some period of time. In order for the average speed to be 60, the car must have been going faster than 60 for some period of time as well.