Math 180: Calculus I **Fall 2014**

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1. Evaluate the following limits

   (a) $\lim_{x\to e} \frac{\ln x - 1}{x-e}$
   SOLUTION: If we were to plug in e we see that this takes the form $\frac{0}{0}$, therefore we may use L'Hôpital's Rule
   $$\lim_{x\to e} \frac{\ln x - 1}{x-e} = \lim_{x\to e} \frac{1/x}{1} = \frac{1}{e}$$

   (b) $\lim_{u\to \pi/4} \frac{\tan u - \cot u}{u-\pi/4}$
   SOLUTION: If we were to plug in $\pi/4$ we would get the form $\frac{\infty}{\infty}$, so we may use L'Hôpital's Rule
   $$\lim_{u\to \pi/4} \frac{\tan u - \cot u}{u-\pi/4} = \lim_{u\to \pi/4} \frac{\sec^2 u + \csc^2 u}{1} = 4$$

   (c) $\lim_{x\to \infty} \frac{3x^4 - x^2}{6x^4+12}$
   SOLUTION: If we were to "plug in " ∞, we would get the form $\frac{\infty}{\infty}$, so we may use L'Hôpital's Rule
   $$\begin{aligned}\lim_{x\to\infty} \frac{3x^4-x^2}{6x^4+12} &= \lim_{x\to\infty} \frac{12x^3-2x}{24x^3} && \text{use L'Hôpital's Rule}\\ &= \lim_{x\to\infty}\frac{36x^2-2}{72x^2} && \text{use L'Hôpital's Rule again}\\ &= \lim_{x\to\infty}\frac{72x}{144x} && \text{use L'Hôpital's Rule again}\\ &= \frac{1}{2}\end{aligned}$$

   (d) $\lim_{x\to \pi/2} \frac{2\tan x}{\sec^2 x}$
   SOLUTION: If we were to plug in $\pi/2$ we would have the form $\frac{\infty}{\infty}$, so we may use L'Hôpital's Rule.
   $$\lim_{x\to \pi/2} \frac{2\tan x}{\sec^2 x} = \lim_{x\to \pi/2} \frac{2\sec^2 x}{2\sec x(\sec x \tan x} = \lim_{x\to \pi/2} \frac{1}{\tan x} = 0$$

   (e) $\lim_{x\to 0} x\csc x$
   SOLUTION: If we plug 0 in, we get the form $0\cdot\infty$, so we must first recast this as a fraction. We could use either of these facts:
   $$x = \frac{1}{1/x} \text{ (if } x\neq 0\text{), or } \csc x = \frac{1}{\sin x}$$
   We would prefer to keep x in the numerator, because its derivative is simply 1, whereas the derivative of $1/x$ in the denominator is $-1/x^2$. So let's use the second substitution.
   $$\lim_{x\to 0} x\csc x = \lim_{x\to 0}\frac{x}{\sin x}$$
   Now we may use L'Hôpital's Rule.
   $$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\cos x} = 1$$

(f) $\lim_{x\to\infty} x - \sqrt{x^2-1}$
SOLUTION: If we "plug in" ∞ we have the form $\infty - \infty$, we must first factor out something to try to put it in the form of $\infty \cdot 0$. If we factor out x^2 from the expression under the square root we get

$$\lim_{x\to\infty} x - \sqrt{x^2-1} = \lim_{x\to\infty} x - x\sqrt{1-1/x^2} = \lim_{x\to\infty} x(1-\sqrt{1-1/x^2})$$

If we make a substitution that $t = \frac{1}{x}$, and we note

$$\lim_{x\to\infty} t = \lim_{x\to\infty} \frac{1}{x} = 0$$

So the original limit becomes

$$\lim_{x\to\infty} x(1-\sqrt{1-1/x^2} = \lim_{t\to 0} \frac{1-\sqrt{1-t^2}}{t}$$

which is of the form $\frac{0}{0}$, so we may use L'Hôpital's Rule.

$$\lim_{t\to 0} \frac{1-\sqrt{1-t^2}}{t} = \lim_{t\to 0} -\frac{1}{2}(1-t^2)^{-1/2}(-2t) = 0$$

(g) $\lim_{x\to 0^+} x^{2x}$
SOLUTION: To handle limits of the form $f(x)^{g(x)}$, we first find L, the limit of the logarithm.

$$L = \lim_{x\to 0^+} 2x\ln x$$

We can write $x = \frac{1}{1/x}$ to recast this as a fraction.

$$L = \lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

This is of the form $\frac{\infty}{\infty}$, so we may use L'Hôpital's Rule.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

Since $L = 0$, the original limit is $e^L = 1$.

(h) $\lim_{x\to 0}(1+4x)^{3/x}$
SOLUTION: Again we must take the limit of the log and find L.

$$L = \lim_{x\to 0} \frac{3\ln(1+4x)}{x}$$

Which is of the form $\frac{0}{0}$, so we may use L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{3\ln(1+4x)}{x} = \lim_{x\to 0} \frac{12}{1+4x} = 12$$

The the original limit is e^12.

(i) $\lim_{\theta\to\pi/2^-}(\tan\theta)^{\cos\theta}$
SOLUTION: Again we must take the limit of the log.

$$L = \lim_{\theta\to\pi/2^-} \cos\theta\ln(\tan\theta)$$

This takes the form of $0 \cdot \infty$, we we recast it as a fraction.

$$\lim_{\theta\to\pi/2^-} \cos\theta \ln(\tan\theta) = \lim_{\theta\to\pi/2^-} \frac{\ln(\tan\theta)}{\sec\theta}$$

This takes the form $\frac{\infty}{\infty}$, so we may use L'Hôpital's Rule.

$$\lim_{\theta\to\pi/2^-} \frac{\ln(\tan\theta)}{\sec\theta} = \lim_{\theta\to\pi/2^-} \frac{\sec^2\theta}{\tan\theta\cdot\sec\theta\tan\theta} = \lim_{\theta\to\pi/2^-} \frac{\sec\theta}{\tan^2\theta} = \lim_{\theta\to\pi/2^-} \frac{\cos\theta}{\sin^2\theta} = 0$$

So the original limit is $e^0 = 1$

2. Compare the growth rates of the following functions

   (a) $x^{10}; e^{0.01x}$
   SOLUTION: To compare the growth rates, we evaluate the following limit

   $$\lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}}$$

   By applying L'Hôpital's Rule 10 times we finally get

   $$\lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}} = \lim_{x\to\infty} \frac{10!}{(0.01)^10e^{0.01x}} = 0$$

   So the exponential function has a greater growth rate.

   (b) $\ln\sqrt{x}; \ln^2 x$
   SOLUTION: To compare growth rates, we evaluate the following limit

   $$\lim_{x\to\infty} \frac{\ln\sqrt{x}}{\ln^2 x}$$

   Before using L'Hôpital's Rule, let's use log properties to re-write the numerator

   $$\lim_{x\to\infty} \frac{\ln\sqrt{x}}{\ln^2 x} = \lim_{x\to\infty} \frac{\frac{1}{2}\ln x}{\ln x\cdot\ln x} = \lim_{x\to\infty} \frac{1}{2\ln x} = 0$$

   So $\ln^2 x$ has a greater growth rate.

3. Evaluate this limit, which appeared in L'Hôpital's book.

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

SOLUTION: If we plug in a, we get the form $\frac{0}{0}$, so we apply the rule.

$$\begin{aligned}
\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} &= \lim_{x\to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - \frac{a}{3}(a^2x)^{-2/3}(a^2)}{-\frac{1}{4}(ax^3)^{-37/4}(3ax^2)} \\
&= \frac{\frac{1}{2}(2a^4 - a^4)^{-1/2}(2a^3 - 4a^3) - \frac{a^3}{3}(a^3)^{-2/3}}{-\frac{1}{4}(a^4)^{-3/4}(3a^3)} \\
&= \frac{\frac{1}{2}(a^{-2}(-2a^3) - \frac{a^3}{3}a^{-2}}{-\frac{1}{4}a^{-3}(3a^3)} \\
&= \frac{-a^{-6} - \frac{a^{-6}}{3}}{-\frac{3}{4}} \\
&= \frac{-\frac{4}{3}a^{-6}}{-\frac{3}{4}} \\
&= \frac{16}{9a^6}
\end{aligned}$$

4. Consider the following limit

$$\lim_{x\to\infty}\frac{\sqrt{ax+b}}{\sqrt{cx+d}}$$

where a, b, c, d are all positive real numbers. What happens when L'Hôpital's rule is used? How else can the limit be found?
SOLUTION: When L'Hôpital's rule is used, we just get

$$\lim_{x\to\infty}\frac{\sqrt{cx+d}}{\sqrt{ax+b}}$$

Which is basically where we started. But we forgot that this may be written

$$\lim_{x\to\infty}\frac{\sqrt{ax+b}}{\sqrt{cx+d}} = \lim_{x\to\infty}\sqrt{\frac{ax+b}{cx+d}} = \sqrt{\lim_{x\to\infty}\frac{ax+b}{cx+d}} = \sqrt{\frac{a}{c}}$$

5. Find all antiderivatives
   (a) $g(x) = 11x^{10}$
   SOLUTION: $G(x) = x^{11}$
   (b) $f(x) = -4\cos(4x)$
   SOLUTION: $F(x) = -\sin(4x)$
   (c) $f(y) = \frac{-2}{y^3}$
   SOLUTION: $F(y) = \frac{1}{y^2}$
6. Solve the indefinite integrals
   (a) $\int(3x^5 - 5x^9)dx$
   SOLUTION: $\int(3x^5 - 5x^9)dx = \frac{3}{6}x^6 - \frac{5}{10}x^{10} + C$
   (b) $\int(\sec^2 - 1)dx$
   SOLUTION: $\int(\sec^2 - 1)dx = \tan x - x + C$
   (c) $\int \frac{3}{4+x^2}dx$
   SOLUTION: We recognize the pattern of $\tan^{-1}$, but it's not quite right. First we should factor out $\frac{3}{4}$

$$\int\frac{3}{4+x^2}dx = \frac{3}{4}\int\frac{1}{1+x^2/4}dx$$

   Now we can make a substitution, letting $u = x/2$, so $du = dx/2$ which means $dx = 2du$

$$\frac{3}{4}\int\frac{1}{1+x^2/4}dx = \frac{3}{4}\int\frac{2}{1+u^2}du = \frac{3}{2}\tan^{-1}u + C = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

7. Solve for the antiderivative using the initial conditions
   (a) $f(t) = \sec^2 t, F(\pi/4) = 1$
   SOLUTION: $F(t) = \tan t + C$, so plugging in $\pi/4$ we have $1 = 1 + C$, so $C = 0$. The final answer is $F(t) = \tan t$.
   (b) $g'(x) = 7x(x^6 - \frac{1}{7}), g(1) = 24$
   SOLUTION: Rewrite $g'(x) = 7x^7 - x$. So $g(x) = \frac{7}{8}x^8 - \frac{1}{2}x^2 + C$. Plugging in 1 we have $24 = \frac{7}{8} - \frac{1}{2} + C$ so $C = 23.625$. Finally the answer is $g(x) = \frac{7}{8}x^8 - \frac{1}{2}x^2 + 23.625$
   (c) $F''(x) = \cos x, F'(0) = 3, F(\pi) = 4$
   SOLUTION: $F'(x) = \sin x + C$, and plugging in 0 we get $3 = C$. So we have $F'(x) = \sin x + 3$. Then $F(x) = -\cos x + 3x + K$, (so we aren't reusing letters). Using the initial condition, we get $4 = 1 + 3\pi + K$, so $K = 3 - 3\pi$. Thus $F(x) = -\cos x + 3x + 3 - 3\pi$.