1. Evaluate the following limits

(a) \( \lim_{x \to e} \frac{\ln x - 1}{x - e} \)

**SOLUTION:** If we were to plug in \( e \) we see that this takes the form \( \frac{0}{0} \), therefore we may use L'Hôpital's Rule

\[
\lim_{x \to e} \frac{\ln x - 1}{x - e} = \lim_{x \to e} \frac{1/x}{1} = \frac{1}{e}
\]

(b) \( \lim_{u \to \pi/4} \frac{\tan u - \cot u}{u - \pi/4} \)

**SOLUTION:** If we were to plug in \( \pi/4 \) we would get the form \( \frac{\infty}{\infty} \), so we may use L'Hôpital's Rule

\[
\lim_{u \to \pi/4} \frac{\tan u - \cot u}{u - \pi/4} = \lim_{u \to \pi/4} \frac{\sec^2 u + \csc^2 u}{1} = 4
\]

(c) \( \lim_{x \to \infty} \frac{3x^4 - x^2}{6x^4 + 12} \)

**SOLUTION:** If we were to “plug in ” \( \infty \), we would get the form \( \frac{\infty}{\infty} \), so we may use L'Hôpital's Rule

\[
\lim_{x \to \infty} \frac{3x^4 - x^2}{6x^4 + 12} = \lim_{x \to \infty} \frac{12x^3 - 2x}{24x^3} = \lim_{x \to \infty} \frac{36x^2 - 2}{72x^2} = \lim_{x \to \infty} \frac{72x}{144x} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}
\]

(d) \( \lim_{x \to \pi/2} \frac{2\tan x}{\sec^2 x} \)

**SOLUTION:** If we were to plug in \( \pi/2 \) we would have the form \( \frac{\infty}{\infty} \), so we may use L'Hôpital's Rule.

\[
\lim_{x \to \pi/2} \frac{2\tan x}{\sec^2 x} = \lim_{x \to \pi/2} \frac{2\sec^2 x}{2\sec x(\sec x \tan x)} = \lim_{x \to \pi/2} \frac{1}{\tan x} = 0
\]

(e) \( \lim_{x \to 0} x \csc x \)

**SOLUTION:** If we plug 0 in, we get the form \( 0 \cdot \infty \), so we must first recast this as a fraction. We could use either of these facts:

\[
x = \frac{1}{1/x} \quad \text{(if } x \neq 0) \quad \text{or} \quad \csc x = \frac{1}{\sin x}
\]

We would prefer to keep \( x \) in the numerator, because its derivative is simply 1, whereas the derivative of \( 1/x \) in the denominator is \(-1/x^2\). So let’s use the second substitution.

\[
\lim_{x \to 0} x \csc x = \lim_{x \to 0} \frac{x}{\sin x}
\]

Now we may use L'Hôpital’s Rule.

\[
\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\cos x} = 1
\]
(f) \( \lim_{x \to \infty} x - \sqrt{x^2 - 1} \)

**SOLUTION:** If we “plug in” \( \infty \) we have the form \( \infty - \infty \), we must first factor out something to try to put it in the form of \( \infty \cdot 0 \). If we factor out \( x^2 \) from the expression under the square root we get

\[
\lim_{x \to \infty} x - \sqrt{x^2 - 1} = \lim_{x \to \infty} x - x\sqrt{1 - 1/x^2} = \lim_{x \to \infty} x(1 - \sqrt{1 - 1/x^2})
\]

If we make a substitution that \( t = \frac{1}{x} \), and we note

\[
\lim_{x \to \infty} t = \lim_{x \to \infty} \frac{1}{x} = 0
\]

So the original limit becomes

\[
\lim_{x \to \infty} x(1 - \sqrt{1 - 1/x^2}) = \lim_{t \to 0} \frac{1 - \sqrt{1 - t^2}}{t}
\]

which is of the form \( \frac{0}{0} \), so we may use L’Hôpital’s Rule.

\[
\lim_{t \to 0} \frac{1 - \sqrt{1 - t^2}}{t} = \lim_{t \to 0} \frac{-1 - t^2}{-2t} = 0
\]

(g) \( \lim_{x \to 0^+} x^{2x} \)

**SOLUTION:** To handle limits of the form \( f(x)^{g(x)} \), we first find \( L \), the limit of the logarithm.

\[
L = \lim_{x \to 0^+} 2x \ln x
\]

We can write \( x = \frac{1}{1/x} \) to recast this as a fraction.

\[
L = \lim_{x \to 0^+} 2x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}
\]

This is of the form \( \frac{\infty}{\infty} \), so we may use L’Hôpital’s Rule.

\[
\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0
\]

Since \( L = 0 \), the original limit is \( e^L = 1 \).

(h) \( \lim_{x \to 0}(1 + 4x)^{3/x} \)

**SOLUTION:** Again we must take the limit of the log and find \( L \).

\[
L = \lim_{x \to 0} \frac{3 \ln(1 + 4x)}{x}
\]

Which is of the form \( \frac{0}{0} \), so we may use L’Hôpital’s Rule.

\[
\lim_{x \to 0} \frac{3 \ln(1 + 4x)}{x} = \lim_{x \to 0} \frac{12}{1 + 4x} = 12
\]

The the original limit is \( e^L = 12 \).

(i) \( \lim_{\theta \to \pi/2^-} (\tan \theta)^{\cos \theta} \)

**SOLUTION:** Again we must take the limit of the log.

\[
L = \lim_{\theta \to \pi/2^-} \cos \theta \ln(\tan \theta)
\]
This takes the form of $0 \cdot \infty$, we we recast it as a fraction.

$$\lim_{\theta \to \pi/2^-} \cos \theta \ln(\tan \theta) = \lim_{\theta \to \pi/2^-} \frac{\ln(\tan \theta)}{\sec \theta}$$

This takes the form $\infty \cdot \infty$, so we may use L'Hôpital's Rule.

$$\lim_{\theta \to \pi/2^-} \frac{\ln(\tan \theta)}{\sec \theta} = \lim_{\theta \to \pi/2^-} \frac{\tan \theta \cdot \sec \theta \tan \theta - \sec^2 \theta}{\tan^2 \theta \cdot \sin^2 \theta} = 0$$

So the original limit is $e^0 = 1$

2. Compare the growth rates of the following functions

(a) $x^{10}, e^{0.01x}$

**SOLUTION:** To compare the growth rates, we evaluate the following limit

$$\lim_{x \to \infty} \frac{x^{10}}{e^{0.01x}}$$

By applying L'Hôpital’s Rule 10 times we finally get

$$\lim_{x \to \infty} \frac{x^{10}}{e^{0.01x}} = \lim_{x \to \infty} \frac{10!}{(0.01)^{10} e^{0.01x}} = 0$$

So the exponential function has a greater growth rate.

(b) $\ln \sqrt{x}, 2x$

**SOLUTION:** To compare growth rates, we evaluate the following limit

$$\lim_{x \to \infty} \frac{\ln \sqrt{x}}{\ln^2 x}$$

Before using L'Hôpital’s Rule, let’s use log properties to re-write the numerator

$$\lim_{x \to \infty} \frac{\ln \sqrt{x}}{\ln^2 x} = \lim_{x \to \infty} \frac{\frac{1}{2} \ln x}{\ln x \cdot \ln x} = \lim_{x \to \infty} \frac{1}{2 \ln x} = 0$$

So $\ln^2 x$ has a greater growth rate.

3. Evaluate this limit, which appeared in L’Hôpital’s book.

$$\lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a \sqrt{a^2x}}{a - \sqrt{ax^3}}$$

**SOLUTION:** If we plug in $a$, we get the form $\frac{0}{0}$, so we apply the rule.

$$\lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a \sqrt{a^2x}}{a - \sqrt{ax^3}} = \lim_{x \to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(-2a^3 - 4x^3) - \frac{2}{3}(a^2x)^{-3/2}(a^2)}{-\frac{1}{2}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{1}{2} \left( 2a^4 - a^4 \right)^{-1/2} \left( 2a^3 - 4a^3 \right) - \frac{a^3}{3} (a^3)^{-2/3}$$

$$= \frac{1}{2} \left( a^4 \right)^{-1/2} - \frac{a^3}{3} a^3$$

$$= -a^6 - a^6$$

$$= -4a^6$$

$$= \frac{16}{9a^6}$$
4. Consider the following limit
\[
\lim_{{x \to \infty}} \frac{\sqrt{ax + b}}{\sqrt{cx + d}}
\]
where \(a, b, c, d\) are all positive real numbers. What happens when L'Hôpital’s rule is used? How else can the limit be found?

**SOLUTION:** When L'Hôpital’s rule is used, we just get
\[
\lim_{{x \to \infty}} \frac{\sqrt{cx + d}}{\sqrt{ax + b}}
\]
Which is basically where we started. But we forgot that this may be written
\[
\lim_{{x \to \infty}} \frac{\sqrt{ax + b}}{\sqrt{cx + d}} = \lim_{{x \to \infty}} \frac{ax + b}{cx + d} = \lim_{{x \to \infty}} \frac{ax + b}{c} = \sqrt{\frac{a}{c}}
\]

5. Find all antiderivatives
(a) \(g(x) = 11x^{10}\)

**SOLUTION:** \(G(x) = x^{11}\)

(b) \(f(x) = -4 \cos(4x)\)

**SOLUTION:** \(F(x) = -\sin(4x)\)

(c) \(f(y) = \frac{-2}{y^7}\)

**SOLUTION:** \(F(y) = \frac{1}{y^6}\)

6. Solve the indefinite integrals
(a) \(\int (3x^5 - 5x^9)\,dx\)

**SOLUTION:** \(\int (3x^5 - 5x^9)\,dx = \frac{3}{5}x^6 - \frac{5}{10}x^{10} + C\)

(b) \(\int (\sec^2 - 1)\,dx\)

**SOLUTION:** \(\int (\sec^2 - 1)\,dx = \tan x - x + C\)

(c) \(\int \frac{3}{4 + x^2}\,dx\)

**SOLUTION:** We recognize the pattern of \(\tan^{-1}\), but it’s not quite right. First we should factor out \(\frac{3}{4}\)
\[
\int \frac{3}{4 + x^2}\,dx = \frac{3}{4} \int \frac{1}{1 + u^2}\,du
\]
Now we can make a substitution, letting \(u = x/2\), so \(du = dx/2\) which means \(dx = 2du\)
\[
\frac{3}{4} \int \frac{1}{1 + u^2}\,du = \frac{3}{4} \int \frac{2}{1 + u^2}\,du = \frac{3}{2} \tan^{-1} u + C = \frac{3}{2} \tan^{-1} \left( \frac{x}{2} \right) + C
\]

7. Solve for the antiderivative using the initial conditions
(a) \(f(t) = \sec^2 t, F(\pi/4) = 1\)

**SOLUTION:** \(F(t) = \tan t + C\), so plugging in \(\pi/4\) we have \(1 = 1 + C\), so \(C = 0\). The final answer is \(F(t) = \tan t\).

(b) \(g'(x) = 7x(x^6 - \frac{1}{2}), g(1) = 24\)

**SOLUTION:** Rewrite \(g'(x) = 7x^7 - x\). So \(g(x) = \frac{7}{8}x^8 - \frac{1}{2}x^2 + C\). Plugging in 1 we have
\[
24 = \frac{7}{8} - \frac{1}{2} + C \quad \text{so} \quad C = 23.625
\]
Finally the answer is \(g(x) = \frac{7}{8}x^8 - \frac{1}{2}x^2 + 23.625\)

(c) \(F''(x) = \cos x, F'(0) = 3, F(\pi) = 4\)

**SOLUTION:** \(F'(x) = \sin x + C\), and plugging in 0 we get \(3 = C\). So we have \(F'(x) = \sin x + 3\).
Then \(F(x) = -\cos x + 3x + K\), (so we aren’t reusing letters). Using the initial condition, we get
\[
4 = 1 + 3\pi + K \quad \text{so} \quad K = 3 - 3\pi
\]
Thus \(F(x) = -\cos x + 3x + 3 - 3\pi\).