Math 180: Calculus I **Fall 2014**

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1. Evaluate the following definite integrals

   (a) $\int_0^2 4x^3dx$
   SOLUTION:

   $$\int_0^2 4x^3dx = x^4\Big|_0^2 = 2^4 - 0 = 16$$

   (b) $\int_0^{\pi/4} 2\cos x dx$
   SOLUTION:

   $$\int_0^{\pi/4} 2\cos x dx = 2\sin x|_0^{\pi/4} = 2\sin(\frac{\pi}{4}) - 2\sin(0) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$$

   (c) $\int_{-2}^2 (x^2-4)dx$
   SOLUTION:

   $$\int_{-2}^2 (x^2-4)dx = \frac{1}{3}x^3 - 4x\Big|_{-2}^2 = \frac{8}{3} - 8 - \left(\frac{-8}{3}+8\right) = -\frac{32}{3}$$

   (d) $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$
   SOLUTION:

   $$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\big|_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

   (e) $\int_0^1 10e^{2x}dx$
   SOLUTION:

   $$\int_0^1 10e^{2x}dx = \frac{10}{2}e^{2x}\Big|_0^1 = 5e^2 - 5e^0 = 5e^2 - 5$$

   (f) $\int_1^3 \frac{3}{t}dt$
   SOLUTION:

   $$\int_1^3 \frac{3}{t}dt = 3\ln|t||_1^3 = 3ln3 - 3\ln 1 = 3\ln 3$$

2. Find the area of the region bounded by the x-axis, and $y = 4 - x^2$.
   The x intercepts are at $x = \pm 2$ so we want to evaluate the integral

   $$\int_{-2}^2 (4-x^2)dx$$

   But this is just the negative of 1c, so this will come to $-(-\frac{32}{3}) = \frac{32}{3}$.

3. Simplify the following expressions using the FTC.

(a) $\frac{d}{dx}\int_3^x (t^2+t+1)dt$
SOLUTION:

$$\frac{d}{dx}\int_3^x (t^2+t+1)dt = \frac{d}{dx}\left(\frac{1}{3}t^3+\frac{1}{2}t^2+t\Big|_3^x\right) = \frac{d}{dx}\left(\frac{1}{3}x^3+\frac{1}{2}x^2+x-\left(\frac{27}{3}+\frac{9}{2}+3\right)\right) = x^2+x+1$$

Although we could have jumped immediately to the final answer using the fundamental theorem of calculus.

(b) $\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2+1}$

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2+1} &= -\frac{d}{dx}\int_{10}^{x^2} \frac{dz}{z^2+1} && \text{swapping the bounds changes the sign}\\ &= \frac{1}{(x^2)^2+1}(2x) && \text{by the FTC}\end{aligned}$$

(c) $\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln t^2 dt$
SOLUTION:

$$\begin{aligned}\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln t^2 dt &= \frac{d}{dx}\left(\int_{e^x}^{1} \ln t^2 dt + \int_1^{e^{2x}} \ln t^2 dt\right) && \text{You may split the integral anywhere on the domain}\\ &= \frac{d}{dx}\int_{e^x}^{1} \ln t^2 dt + \frac{d}{dx}\int_1^{e^{2x}} \ln t^2 dt && \text{swap bounds changes sign}\\ &= -\frac{d}{dx}\int_1^{e^x} \ln t^2 dt + \frac{d}{dx}\int_1^{e^{2x}} \ln t^2 dt && \text{distribute differential operator}\\ &= -\ln(e^x)^2 + \ln(e^{2x})^2 && \text{by FTC}\end{aligned}$$

4. Evaluate the following definite integrals

(a) $\frac{1}{2}\int_0^{\ln 2} e^x dx$
SOLUTION:

$$\frac{1}{2}\int_0^{\ln 2} e^x dx = \frac{1}{2}\left. e^x\right|_0^{\ln 2} = \frac{1}{2}\left(e^{\ln 2}-e^0\right) = \frac{1}{2}(2-1) = \frac{1}{2}$$

(b) $\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$
SOLUTION:

$$\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\Big|_{\sqrt{2}}^2 = \sec^{-1}(2) - \sec^{-1}(\sqrt{2}) = \frac{\pi}{3}-\frac{\pi}{4} = \frac{\pi}{12}$$

5. What value of $b > -1$ maximizes the integral

$$\int_{-1}^b x^2(3-x)dx$$

SOLUTION: The integral, call it $A(b)$ is

$$A(b) = \int_{-1}^b x^2(3-x)dx = \int_{-1}^b (3x^2-x^3)dx = x^3 - \frac{x^4}{4}\Big|_{-1}^b = b^3 - \frac{b^2}{4} - 1 - \frac{1}{4}$$

The first derivative of this with respect to b is simply $A'(b) = b^2(3-b)$, (we could have figured that out without doing any antiderivatives) which has zeroes at 0 and 3. These are the critical points. We can plug in these into the function to find which maximizes the definite integral. We'll find that $A(3) = 27 - \frac{9}{4} - \frac{5}{4} = 24$ is the maximum, so $b = 3$ maximizes the integral.

6. Suppose f is a continuous function of t on $[0, \infty)$ and $A(x)$ is the net area of the region bounded by the graph of f and the t-axis on $[0, x]$. Show that the local maxima and minima of A occur at the zeroes of f. Verify this with $f(t) = t^2 - 10t$.
   SOLUTION: $A(x) = \int_0^x f(t)dt$. The derivative, $A'(x) = \frac{d}{dx} \int_0^x f(t)dt = f(x)$ by the fundamental theorem of calculus. The critical points of A will be among the zeroes of f (since f is continuous, this means that the derivative of A is continuous, so it is defined everywhere, it has no cusps or corners). So Local minima and maxima of the area must be zeroes of f.

   For example, let $A(x) = \int_0^x (t^2 - 10t)dt = \frac{1}{3}x^3 - 5x$. If we want to find the critical points of this function, we take its derivative. $A'(x) = x^2 - 10x = x(x - 10)$. So its critical points are 0 and 10 which are a local minimum and maximum respectively.