Math 180: Calculus I **Fall 2014**

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1. Use symmetry to evaluate these integrals

   (a) $\int_{-\pi/4}^{\pi/4} \cos x dx$
   SOLUTION: Since the cosine function is an evan function, and the interval is syummetric to the y-axis, it suffices to double the integral on $[0, \pi/4]$.

   $$\int_{-\pi/4}^{\pi/4} \cos x dx = 2\int_{0}^{\pi/4} \cos x dx = 2 \sin x|_0^{\pi/4} = 2\frac{\sqrt{2}}{2} - 0 = \sqrt{2}$$

   (b) $\int_{-10}^{10} \frac{x}{\sqrt{200-x^2}} dx$ **SOLUTION:** Since the numerator is an odd function and the denominator is an even function, this function is odd. Since the interval is symmetric to the y-axis, the definite integral will be zero.

   (c) $\int_0^{2\pi} \sin x dx$
   SOLUTION: The function is an odd function, and since it is periodic, we may argue that

   $$\int_0^{2\pi} \sin x dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} \sin x dx = \int_0^{\pi} \sin x dx + \int_{-\pi}^{0} \sin x dx = \int_{-\pi}^{\pi} \sin x dx$$

   If we write it like this, the interval is symmetric to the y-axis, and we can justify that the definite integral will evaluate to zero.

2. Find the average value of the following functions on the interval given

   (a) $f(x) = 1/x; [1, e]$
   SOLUTION: The average value of the function is found by evaluating

   $$\frac{1}{e-1}\int_1^e \frac{1}{x} dx = \frac{1}{e-1}\left[\ln x\right]_1^e = \frac{\ln e - \ln 1}{e-1} = \frac{1}{e-1}$$

   (b) $f(x) = x(1-x); [0, 1]$
   SOLUTION:

   $$\frac{1}{1-0}\int_0^1 (x - x^2) dx = \frac{1}{1}\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. Find the appropriate point in the interval where the function equals its average value.

   (a) $f(x) = e^x; [0, 2]$
   SOLUTION: The everage value is

   $$\frac{1}{2}\int_0^2 e^x dx = \frac{1}{2}\left[e^x\right]_0^2 = \frac{e^2-1}{2}$$

   To find the point where the function equals this, just set the function equal and solve for x.

   $$e^x = \frac{e^2-1}{2}$$
   $$x = \ln\left(\frac{e^2-1}{2}\right)$$

(b) $f(x) = 1 - |x|; [-1, 1]$
SOLUTION: The average value is found using symmetry

$$\frac{1}{2}\int_{-1}^{1}(1-|x|)dx = \frac{1}{2}2\int_{0}^{1}(1-x)dx = \left[x - \frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

. Setting the function equal we get:

$$\begin{aligned} 1 - |x| &= \frac{1}{2} \\ |x| &= \frac{1}{2} \\ x &= \pm\frac{1}{2} \end{aligned}$$

4. Show that the area of a segment of a parabola is 4/3 that of the inscribed triangle of greatest area. Specificaly, show that the area bounded by $y = a^2 - x^2$ and the x-axis is 4/3 the area of the triangle with vertices at $(\pm a, 0)$ and $(0, a^2)$. Let $a > 0$ be an arbitrary constant.
SOLUTION: The parabola will intersect the x-axis when $0 = a^2 - x^2$ or $x = \pm a$. The area under the parabola is

$$\int_{-a}^{a}(a^2 - x^2)dx = 2\int_{0}^{a}(a^2 - x^2)dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = 2\left(a^3 - \frac{1}{3}a^3\right) = \frac{4a^3}{3}$$

The area under a triangle as described (with base $2a$ and height a^2) is

$$\frac{1}{2}2aa^2 = a^3$$

We can see that the ratio of their areas is $\frac{4}{3}$.

5. Use a change of variables (substitution) to find the following integrals

(a) $\int 2x(x^2 - 1)^{99}dx$
SOLUTION: Let $u = x^2 - 1$, $du = 2xdx$ we have

$$\begin{aligned} \int 2x(x^2-1)^{99}dx &= \int u^{99}du \\ &= \frac{1}{100}u^{100} + C \\ &= \frac{1}{100}(x^2-1)^{100} + C \end{aligned}$$

(b) $\int x^3(x^4 + 16)^6dx$
SOLUTION: Let $u = x^4 + 16$, $du = 4x^3dx$, so $x^3dx = \frac{1}{4}du$. The substitution can be made

$$\begin{aligned} \int x^3(x^4+16)^6dx &= \int \frac{1}{4}u^6du \\ &= \frac{7}{4}u^7 + C \\ &= \frac{7}{4}(x^4+16)^7 + C \end{aligned}$$

(c) $\int 2x\sin(x^2)dx$
Let $u = x^2$, $du = 2xdx$, we have

(d) $\int \frac{x^2}{(x+1)^4} dx$
SOLUTION: Letting $u = x + 1$, $du = dx$. This may not seem to make a big difference, but it will help. We need to use the substitution $x = u - 1$.

$$\begin{aligned}
\int \frac{x^2}{(x+1)^4} dx &= \int \frac{(u-1)^2}{u^4} du \\
&= \int \frac{u^2 - 2u + 1}{u^4} du \\
&= \int \left(\frac{u^2}{u^4} - 2\frac{u}{u^4} + \frac{1}{u^4} \right) du \\
&= \int \left(u^{-2} - 2u^{-3} + u^{-4} \right) du \\
&= -u^{-1} - \frac{2}{-2} u^{-2} + \frac{1}{-3} u^{-3} + C
\end{aligned}$$

(e) $\int (x+1)\sqrt{3x+2} dx$
SOLUTION: Let $u = 3x + 2$, so $du = 3dx$ or $dx = \frac{1}{3} du$. We may re-write the u substitution $u + 1 = 3x + 3$ or $x + 1 = \frac{1}{3}(u + 1)$. Then we have all we need to make some substitutions.

$$\begin{aligned}
\int (x+1)\sqrt{3x+2} dx &= \int \frac{1}{3}(u+1)\sqrt{u}\frac{1}{3} du \\
&= \frac{1}{9} \int (u^{3/2} + u^{1/2}) du \\
&= \frac{1}{9} \left(\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + C \\
&= \frac{1}{9} \left(\frac{2}{5} (3x+2)^{5/2} + \frac{2}{3} (3x+2)^{3/2} \right) + C
\end{aligned}$$

(f) $\int_0^1 2x(4 - x^2) dx$
SOLUTION: First, let $u = 4 - x^2$, so $du = -2xdx$, then $2xdx = -du$. Since this is a definite integral, we should convert the bounds from x values to u values using the u substitution rule. The lower bound becomes $4 - 0^2 = 4$, the upper bound becomes $4 - 1^2 = 3$

$$\begin{aligned}
\int_0^1 2x(4 - x^2) dx &= \int_4^3 -udu \\
&= -\left[\frac{1}{2} u^2 \right]_4^3 \\
&= -\frac{1}{2}(9 - 16) \\
&= \frac{7}{2}
\end{aligned}$$

(g) $\int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta$
SOLUTION: Here we let $u = \sin \theta$ so $du = \cos \theta d\theta$. The lower bound maps to $\sin 0 = 0$, the

upper bound maps to $\sin(\pi/2) = 1$.

$$\begin{aligned}\int_0^{\pi/2} \sin^2\theta\cos\theta d\theta &= \int_0^1 u^2 du \\ &= \left[\frac{1}{3}u^3\right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

(h) $\int_0^4 \frac{p}{\sqrt{9+p^2}}dp$
SOLUTION: Here if we let $u = 9+p^2$, $du = 2pdp$ so $pdp = \frac{1}{2}du$. The lower bound is $9+0^2 = 9$, the upper bound is $9+4^2 = 25$.

$$\begin{aligned}\int_0^4 \frac{p}{\sqrt{9+p^2}}dp &= \int_9^{25} \frac{1}{2}\frac{1}{\sqrt{u}}du \\ &= \frac{1}{2}\int_9^{25} u^{-1/2}du \\ &= \frac{1}{2}\left[2u^{1/2}\right]_9^{25} \\ &= 5-3 \\ &= 2\end{aligned}$$