Example 4.1 Is the following function continuous at $a$?

1. $f(x) = \frac{2x^2+3x+1}{x^2+5x}; a = 5$
2. $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ \frac{3}{x-1} & \text{if } x = 1 \end{cases}; a = 1$

1. This is a rational function, and rational functions are continuous at all points in the domain. The thing to check is whether 5 is in the domain or not. Does 5 kill us in the denominator? No, it doesn’t cause us to divide by zero, so it’s in the domain - the answer is yes.

2. Check the definition for continuity:
   a) is the function defined at $a$? Yes, $f(a) = 3$.
   b) does the limit $L$ exist at $a$? Yes,
   
   $$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = 2$$

   c) does the $f(a) = L$? No! The function is not continuous at $a$.

Example 4.2 On what intervals are the following functions continuous?

1. $f(x) = \frac{x^3+6x+17}{x^2-9}$
2. $f(x) = \frac{1}{x^2-3}$

1. Again, since $f$ is rational, we just have to give all intervals of the domain. The denominator factors into $(x - 3)(x + 3)$ so the domain is all real numbers except 3 and -3. Therefore, the intervals where $f$ is continuous are

   $$(-\infty, -3), (-3, 3), (3, \infty).$$

2. Similarly, the denominator factors into $(x - 2)(x + 2)$ so the intervals where $f$ is continuous are

   $$(-\infty, -2), (-2, 2), (2, \infty).$$

Example 4.3 Show that $f(x)$ is not continuous at 1.

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 3x & \text{if } x \geq 1 \end{cases}$$

Is $f$ left continuous or right continuous at 1?
We check the definition of continuity and see where it breaks down.
a) Is \( f(1) \) defined? Yes, \( f(1) = 4 \).
b) Does the limit \( L \) exist as \( x \to 1 \)? No!

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2
\]

but

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 3x = 4.
\]

So the limit does not exist. But because \( \lim_{x \to 1^+} f(x) = f(1) \), We can say that \( f \) is right continuous at 1. (It is not left-continuous though).

**Example 4.4** Does \( f(x) = x \sin(\frac{1}{x}) \) have a removable discontinuity at \( x = 0 \)? Does \( g(x) = \sin(\frac{1}{x}) \)?

\( f(0) \) is not defined, so it is not continuous at 0. To determine if the discontinuity is removable, we check the limit. We’ve already shown (using the squeeze theorem) that

\[
\lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0,
\]

So we may remove the discontinuity by extending the function like so

\[
f^*(x) = \begin{cases} 
x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

\( g(x) \) however, does not behave so nicely. It does indeed have a discontinuity at \( x = 0 \), but \( \lim_{x \to 0} \sin(\frac{1}{x}) \) does not exist - the function oscillates infinitely between -1 and 1 near 0, so the discontinuity is not removable.

**Example 4.5** For a function \( f \), if \( |f| \) is continuous at \( a \) does it mean necessarily that \( f \) is continuous at \( a \)?

This is not true. Though this is true for many many examples, one counterexample is all we need to show it is false. Consider

\[
f(x) = \begin{cases} 
1 & \text{if } x < 0 \\
-1 & \text{if } x \geq 0
\end{cases}
\]

Then \( |f(x)| = 1 \), which is continuous at 0, where the original function is not continuous at 0.