Math 180: Calculus I **Fall 2014**

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Example 4.1 *Is the following function continuous at a?*

1. $f(x) = \frac{2x^2+3x+1}{x^2+5x}; a = 5$
2. $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}; a = 1$

1. This is a rational function, and rational functions are continuous at all points in the domain. The thing to check is whether 5 is in the domain or not. Does 5 kill us in the denominator? No, it doesn't cause us to divide by zero, so it's in the domain - the answer is yes.

2. Check the definition for continuity:
a) is the function defined at a? Yes, $f(a) = 3$.
b) does the limit L exist at a? Yes,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 2$$

c) does the $f(a) = L$? No! The function is not continuous at a.

Example 4.2 *On what intervals are the following functions continuous?*

1. $f(x) = \frac{x^5+6x+17}{x^2-9}$
2. $f(x) = \frac{1}{x^2-4}$

1. Again, since f is rational, we just have to give all intervals of the domain. The denominator factors into $(x-3)(x+3)$ so the domain is all real numbers except 3 and -3. Therefore, the intervals where f is continuous are

$$(-\infty, -3), (-3, 3), (3, \infty).$$

2. Similarly, the denominator factors into $(x-2)(x+2)$ so the intervals where f is continuous are

$$(-\infty, -2), (-2, 2), (2, \infty).$$

Example 4.3 *Show that $f(x)$ is not continuous at 1.*

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 3x & \text{if } x \geq 1 \end{cases}$$

Is f left continuous or right continuous at 1?

We check the definition of continuity and see where it breaks down.
a) Is $f(1)$ defined? Yes, $f(1) = 4$.
b) Does the limit L exist as $x \to 1$? No!

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2$$

but

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 3x = 4.$$

So the limit does not exist. But because $\lim_{x\to 1^+} f(x) = f(1)$, We can say that f is right continuous at 1. (It is not left-continuous though).

Example 4.4 *Does $f(x) = x\sin(\frac{1}{x})$ have a removable discontinuity at $x = 0$? Does $g(x) = \sin(\frac{1}{x})$?*

$f(0)$ is not defined, so it is not continuous at 0. To determine if the discontinuity is removable, we check the limit. We've already shown (using the squeeze theorem) that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0,$$

So we may remove the discontinuity by extending the function like so

$$f^*(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g(x)$ however, does not behave so nicely. It does indeed have a discontinuity at $x = 0$, but $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exist - the function oscillates infinitely between -1 and 1 near 0, so the discontinuity is not removable.

Example 4.5 *For a function f, if $|f|$ is continuous at a does it mean necessarily that f is continuous at a?*

This is not true. Though this is true for many many examples, one counterexample is all we need to show it is false. Consider

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$$

Then $|f(x)| = 1$, which is continuous at 0, where the original function is not continuous at 0.