Math 180: Calculus I **Fall 2014**

September 18

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1. Find the derivatives of the following functions.

   (a) $f(x) = 6x - 2xe^x$
   SOLUTION: Use the sum/difference rule, followed by a product rule

   $$f'(x) = \left(\frac{d}{dx}6x\right) - \left(\frac{d}{dx}\underbrace{2x}_{u}\underbrace{e^x}_{v}\right) = 6 - (u'v + v'u) = 6 - ((2)e^x + (2x)e^x)$$

   (b) $f(x) = (1 + \frac{1}{x^2})(x^2 + 1)$
   SOLUTION: It's easier to write $\frac{1}{x^2} = x^{-2}$. Use product rule again

   $$\begin{aligned}\frac{d}{dx}\underbrace{(1 + x^{-2})}_{u}\underbrace{(x^2 + 1)}_{v} &= u'v + v'u \quad \text{where } u' = -2x^{-3}, v' = 2x. \\ &= (-2x^{-3})(x^2 + 1) + (2x)(1 + x^{-2}) \\ &= -2x^{-1} - 2x^{-3} + 2x + 2x^{-1} \\ &= 2x - 2x^{-3}\end{aligned}$$

   (c) $f(x) = \frac{2e^x - 1}{2e^x + 1}$
   SOLUTION: Use the quotient rule

   $$\begin{aligned}\frac{d}{dx}\frac{2e^x - 1}{2e^x + 1}\begin{matrix}\leftarrow u \\ \leftarrow v\end{matrix} &= \frac{u'v - v'u}{v^2} \quad \text{where } u' = 2e^x, v' = 2e^x \\ &= \frac{(2e^x)(2e^x + 1) - (2e^x)(2e^x - 1)}{(2e^x + 1)^2} \quad \text{Stop here; don't simplify.}\end{aligned}$$

   (d) $f(t) = 2500e^{.075t}$
   SOLUTION: This just follows from the derivative of e^{kx} for some constant k.

   $$f'(t) = (.075)2500e^{.075t} = 1875e^{.075t}$$

   (e) $f(x) = \frac{1}{x^5}$
   SOLUTION: Write it as $f(x) = x^{-5}$ and it becomes easy.

   $$f'(x) = -5x^{-5-1} = -5x^{-6}$$

2. Find the x values such that the slope of $f(x) = xe^{2x}$ is zero.
   SOLUTION: We first find the derivative, $f'(x)$ using the product rule:

   $$\frac{d}{dx}\underbrace{x}_{u}\underbrace{e^{2x}}_{v} = u'v + v'u = (1)(e^{2x}) + (e^{2x})(x) = e^{2x} + xe^{2x}$$

Now we set this equal to zero and solve for x.

$$0 = e^{2x} + xe^{2x}$$
$$0 = e^{2x}(1+2x) \quad \text{and we can divide by } e^{2x} \text{ because it is nonzero for all } x.$$
$$0 = 1 + 2x$$

The only solution is $x = -\frac{1}{2}$.

3. True or false:

   (a) $\frac{d}{dx}(e^5) = 5e$
   SOLUTION: False! e^5 is not an exponential function - it is a constant function (about 148.413), its derivative is zero. Same goes for part b.

   (b) $\frac{d}{dx}(e^5) = e^5$

4. Based on the following table of x values and function/derivative values, evaluate the following:

x	1	2	3	4
$f(x)$	5	4	3	2
$f'(x)$	3	5	2	1
$g(x)$	4	2	5	3
$g'(x)$	2	4	3	1

$$\text{a)}\ \frac{d}{dx}\left(f(x)g(x)\right)\bigg|_{x=1} \qquad \text{b)}\ \frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right)\bigg|_{x=4}$$

a) **SOLUTION:**

$$\frac{d}{dx}\left(f(x)g(x)\right)\bigg|_{x=1} = \left(f'(x)g(x) + g'(x)f(x)\right)\big|_{x=1} = (3)(4) + (2)(5) = 10$$

b) **SOLUTION:** We use the quotient rule, with $u = xf(x)$ and $v = g(x)$.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right)\bigg|_{x=4} &= \left(\frac{u'v - v'u}{v^2}\right)\bigg|_{x=4} \\
&= \left(\frac{(f(x) + xf'(x))g(x) - g'(x)(xf(x))}{g(x)^2}\right)\bigg|_{x=4} \\
&= \frac{(f(4) + 4f'(4))g(4) - g'(4)(4f(4))}{g(4)^2} \\
&= \frac{(2 + 4\cdot 1)3 - (1)(4\cdot 2)}{3^2} \\
&= \frac{10}{9}
\end{aligned}$$

5. Find $f'(x)$, $f''(x)$, and $f'''(x)$.

   (a) $f(x) = \frac{1}{x}$
   SOLUTION: It is much easier to first write $f(x) = x^{-1}$ and use the power rule.

$$\begin{aligned}
f'(x) &= -1x^{-1-1} &&= -x^{-2} \\
f''(x) &= -(-2)x^{-2-1} &&= 2x^{-3} \\
f'''(x) &= (-3)2x^{-3-1} &&= -6x^{-4}
\end{aligned}$$

(b) $f(x) = x^2 e^{3x}$
SOLUTION: We just have to use product rule.

$$\begin{aligned}\frac{d}{dx} f(x) &= \frac{d}{dx} \underbrace{x^2}_{u} \underbrace{e^{3x}}_{v} \\ &= u'v + v'u \\ &= (2x)(e^{3x}) + (3e^{3x})(x^2) \\ &= 2xe^{3x} + 3x^2 e^{3x}\end{aligned}$$

It is helpful when evaluating the second derivative to write $f'(x) = 2xe^3x + 3f(x)$.

$$\begin{aligned}\frac{d}{dx} f'(x) &= \frac{d}{dx} \left((\underbrace{2x}_{u} \underbrace{e^{3x}}_{v}) + 3f(x) \right) \\ &= u'v + v'u + 3f'(x) \\ &= (2)(e^{3x}) + (3e^{3x})(2x) + 3(2xe^{3x} + 3x^2 e^{3x}) \\ &= 2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x} \\ &= 2e^{3x} + 12xe^{3x} + 9x^2 e^{3x}\end{aligned}$$

Again, it may be helpful to write this as $f''(x) = 2e^{3x} + 6f'(x) - 9f(x)$ for the third derivative.

$$\begin{aligned}\frac{d}{dx} f''(x) &= \frac{d}{dx} \left(2e^{3x} + 6f'(x) - 9f(x)\right) \\ &= (3)2e^{3x} + 6f''(x) - 9f'(x) \\ &= 6e^{3x} + 6(2e^{3x} + 12xe^{3x} + 9x^2 e^{3x}) - 9(2xe^{3x} + 3x^2 e^{3x}) \\ &= 6e^{3x} + 12e^{3x} + 72xe^{3x} + 54x^2 e^{3x} - 18xe^{3x} - 27x^2 e^{3x} \\ &= 18e^{3x} + 54xe^{3x} + 27x^2 e^{3x}\end{aligned}$$

6. Find some f and g non-constant functions such that $\frac{d}{dx} f(x)g(x) = f'(x)g'(x)$

SOLUTION: This is very tricky. We wouldn't expect you to get this one, but here's a solution, see if you understand it.
In general, $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ (this is the Product Rule for derivatives). If we find and f and g as desired, then they satisfy the equation

$$f'(x)g(x) + f(x)g'(x) = f'(x)g'(x)$$

Let's put all terms with $f'(x)$ as a factor on the right. We get

$$f(x)g(x) = f'(x)(g'(x) - g(x))$$

For any x such that $g'(x) \neq 0$ we could divide by $g(x)$ and get

$$f(x) = f'(x)\frac{g'(x) - g(x)}{g'(x)} \tag{6.1}$$

What does this tell us about the f we are looking for? Well, for one thing if it is possible that its derivative is a scalar multiple of itself, then it might do the trick. The same goes for g (since we could have done the same thing swapping the places of f and g. So we want functions whose derivatives are scalar multiples of themselves. One such candidate is e^{kx}, whose slope is never equal to zero. Let us try such functions. Assume

$$f(x) = e^{ax}, g(x) = e^{bx} \quad \text{for some } a, b \in \mathbb{R}.$$

Plugging these into equation 6.1, we get

$$e^{ax} = (ae^{ax})\frac{be^{bx} - e^{bx}}{e^{bx}} = ae^{ax}\frac{b-1}{b}$$

Which is true only when

$$a = \frac{b}{b-1}$$

So pick some $b \neq 1$ or 0 and you will get your a. It turns out that $a = b = 2$ works, so one solutions is $f(x) = g(x) = e^{2x}$.