Math 180: Calculus I **Fall 2014**

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For **position** function $s = f(t)$, we have **velocity** at time t is $v(t) = f'(t)$, **speed** at time t is $|v(t)|$, and **acceleration** at t is $a(t) = v'(t) = f''(t)$. **Average cost** $\overline{C}(x) = C(x)/x$. **Marginal cost** is $C'(x)$. For demand function $D = f(p)$, **price elasticity** is $E(p) = \frac{dD}{dp}\frac{p}{D}$. If $-\infty < E(p) < -1$ demand is **elastic**, when $-1 < E(p) < 0$ demand is **inelastic**.

1. Suppose a stone is thrown vertically upward from the edge of a cliff with initial velocity 64 ft/s from a height of 32 ft above the ground. The height h (in ft) of the stone above the ground t seconds after it is thrown is $h = -16t^2 + 64t + 32$.

   (a) Determine the velocity v of the stone after t seconds.
   SOLUTION: $v(t) = h'(t) = -32t + 64$

   (b) When does the stone reach its highest point, and what is its height then?
   SOLUTION: The highest point is when the velocity is zero. Set the velocity equal to zero and solve for t.

   $$0 = -32t + 64 \quad \text{so} \quad t = 2$$

   To find the height at time $t = 2$, evaluate $h(2)$.

   $$h(2) = -16(2)^2 + 64(2) + 32 = 96 \text{ ft.}$$

   (c) When does the stone strike the ground, and what is its velocity at that point?
   SOLUTION: To find the point at which the stone strikes the ground, set the height equal to zero and solve for t.

   $$0 = -16t^2 + 64t + 32 \quad \Rightarrow \quad t = 2 \pm \sqrt{6}$$

   Because $2 - \sqrt{6} < 0$, we can ignore that solution and only use $t = 2 + \sqrt{6}$ seconds. To find the velocity, we evaluate $v(2 + \sqrt{6}$.

   $$v(2 + \sqrt{6}) = -32(2 + \sqrt{6}) + 32 = -32\sqrt{6}\text{ft/sec.}$$

   (d) On what intervals is its speed increasing?
   SOLUTION: It stands to reason that the speed is only increasing as the rock is falling, on the interval $(2, 2 + \sqrt{6})$. We can determine that analytically by finding the derivative of the speed, and see when its derivative is negative.

   $$speed(t) = |v(t)| = \begin{cases} -32t + 64 \text{ if } 0 < t \leq 2 \\ 32t - 64 \text{ if } 2 < t < 2 + \sqrt{6} \end{cases}$$

   And the first derivative is defined piecewise as well

   $$speed'(t) = \begin{cases} -32 \text{ if } 0 < t < 2 \\ 32 \text{ if } 2 < t < 2 + \sqrt{6} \end{cases}$$

   So we have that the speed is increasing on the interval $(2, 2 + \sqrt{6})$, as we expected.

2. For the following (i) find the average cost and marginal cost functions, (ii) Determine the average and marginal cost when $x = a$, and interpret these values

   (a) $C(x) = 1000 + 0.1x, 0 \leq x \leq 5000; a = 2000$
   SOLUTION:

   $$\bar{C}(x) = \frac{C(x)}{x} = \frac{1000}{x} + 0.01, \quad C'(x) = 0.1$$

   So $\bar{C}(a) = \bar{C}(2000) = 1000/2000 + .01 = .51$, $C'(a) = C'(2000) = 0.01$. This means that when the quantity produced is 2000, it costs \$0.51 per item, and the cost to produce one more item is \$0.01.

   (b) $C(x) = -0.01x^2 + 40x + 100, 0 \leq x \leq 1500; a = 1000$
   SOLUTION:

   $$\bar{C}(x) = \frac{C(x)}{x} = -0.01x + 40 + \frac{100}{x}, \quad C'(x) = -0.02x + 40$$

   So $\bar{C}(a) = \bar{C}(1000) = -0.01(1000) + 40 + 100/(1000) = 30.1$ and $C'(a) = C'(1000) = -0.02(1000) + 40 = 20$. This means that when the quantity produced is 1000, it costs \$30.1 per item, and the cost to produce one more item is \$20.00.

3. Compute the elasticity for the exponential demand function $D(p) = ae^{-bp}$, where a and b are positive real numbers. For what prices is the demand elastic? Inelastic?
   SOLUTION: Elasticity $E(p) = \frac{dD}{dp}\frac{p}{D}$ sp

   $$E(p) = (-bae^{-bp})\frac{p}{ae^{-bp}} = -bp$$

   So the price is elastic as long as $-bp < -1$, which is the same as $p > \frac{1}{b}$. It is inelastic if $-1 < -bp < 0$ or $0 < p < \frac{1}{b}$.

4. Show that the demand function $D(p) = a/p^b$, where a and b are positive real numbers, has a constant elasticity for all positive prices.
   SOLUTION: Again, we calculate elasticity, re-writing $D(p) = ap^{-b}$,

   $$E(p) = (-bap^{-b-1})\frac{p}{ap^{-b}} = \frac{-bap^{-b}}{ap^{-b}} = -b$$

   So the elasticity is constant, irrespective of the price p.