Math 180: Calculus I **Fall 2014**

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1. Find the derivative of the following using the chain rule.

   (a) $y = \sin^5 x$
   SOLUTION: $y = \sin(y^5)$, so $g(u) = \sin u$ and $u(x) = x^5$.

   $$y' = g'(u)u'(x) = (\cos u)(5x^4) = 5x^4 \cos(x^5)$$

   (b) $y = \tan(5x^2)$
   SOLUTION: $g(u) = \tan u, u(x) = 5x^2$

   $$y' = g'(u)u'(x) = (\sec^2 u)(10x) = 10x \sec^2(5x^2)$$

   (c) $y = \sin(4\cos x)$
   SOLUTION: $g(u) = \sin u, u(x) = 4\cos x$

   $$y' = g'(u)u'(x) = (\cos u)(-4\sin x) = -4\cos(4\cos x)\cos x$$

   (d) $y = (\sec x + \tan x)^4$
   SOLUTION: $g(u) = u^4, u(x) = \sec x + \tan x$

   $$y' = (4u^3)(\sec x \tan x + \sec^2 x) = 4(\sec x + \tan x)^3(\sec x \tan x + \sec x^2)$$

2. Consider the table

x	1	2	3	4	5
$f'(x)$	-6	-3	8	7	2
$g(x)$	4	1	5	2	3
$g'(x)$	9	7	3	-1	-5

   Let $h(x) = f(g(x))$, and $k(x) = g(g(x))$. Compute the following derivatives:

   (a) $h'(1)$
   SOLUTION: $h'(1) = f'(g(1))g'(1) = f'(4)(9) = (7)(9) = 63$

   (b) $k'(5)$
   SOLUTION: $k'(5) = g'(g(5))g'(5) = g'(3)(-5) = (3)(-5) = -15$

3. Find the derivative using repeated applications of the chain rule:

   (a) $y = \sin(\sin(e^x))$
   SOLUTION: $y = g(u(v(x)))$ where $g(u) = \sin u, u(v) = \sin v, v(x) = e^x$

   $$y' = g'(u(v(x))u'(v(x))v'(x) = \cos(u(v(x))\cos(v(x))e^x = \cos(\sin(e^x))\cos(e^x)e^x$$

(b) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
SOLUTION: $y = g(u(v(x)))$ where $g(u) = \sqrt{u}, u = x + \sqrt{v}, v = x + \sqrt{x}$

$$\begin{aligned} y' &= g'(u)u'(v)v' \\ &= \left(\frac{1}{2\sqrt{u}}\right)\left(1 + \frac{1}{2\sqrt{v}}\left(1 + \frac{1}{2\sqrt{x}}\right)\right) \\ &= \left(\frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\right)\left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)\right) \end{aligned}$$

(c) $y = \left(\frac{x}{x+1}\right)^5$
SOLUTION: $y = u(v(x))$ where $u(v) = v^5, v(x) = \frac{x}{x+1}$

$$y' = 5u^4\left(\frac{(x+1) - (x)}{(x+1)^2}\right) = 5\left(\frac{x}{x+1}\right)^4\left(\frac{1}{(x+1)^2}\right)$$

4. Find the second derivative:

(a) $y = x\cos(x^2)$
SOLUTION: To find the first derivative, we use the product rule, then chain rule to get $\frac{d}{dx}\cos(x^2) = -\sin(x^2)(2x)$.

$$y' = \cos(x^2) + x(-\sin(x^2)(2x) = \cos(x^2) - 2x^2\sin(x^2)$$

The second derivative requires another use of the product rule and chain rule:

$$y'' = -\sin(x^2)(2x) - \left[4x\sin(x^2) + 2x^2(\cos(x^2)(2x))\right]$$

(b) $y = \sqrt{x^2 + 2}$
SOLUTION: Write $y = (x^2 + 2)^{1/2}$

$$y' = \frac{1}{2}(x^2 + 2)^{-1/2}(2x) = 4x(x^2 + 2)^{-1/2}$$

$$y'' = 4(x^2 + 2)^{-1/2} + 4x\left[-\frac{1}{2}(x^2 + 2)^{-3/2}(2x)\right]$$

5. $y''(t) + 2y'(t) + 5y(t) = 0$ is a differential equation. Verify that a solution to the differential equation is

$$y(t) = e^{-t}\left(\sin(2t) - 2\cos(2t)\right).$$

SOLUTION: We need to get the first and second derivative.

$$\begin{aligned} y'(t) &= -e^{-t}\left(\sin(2t) - 2\cos(2t)\right) + e^{-t}(\cos(2t)(2) + 2\sin(2t)(2)) \\ &= e^{-t}(-\sin(2t) + 2\cos(2t) + 2\cos(2t) + 4\sin(2t)) \\ &= e^{-t}(3\sin(2t) + 4\cos(2t)) \\ y''(t) &= -e^{-t}(3\sin(2t) + 4\cos(2t)) + e^{-t}(3\cos(2t)(2) - 4\sin(2t)(2)) \\ &= e^{-t}(-3\sin(2t) - 4\cos(2t) + 6\cos(2t) - 8\sin(2t)) \\ &= e^{-t}(-11\sin(2t) + 2\cos(2t)) \end{aligned}$$

Now we plug $y(t), y'(t)$, and $y''(t)$ in and evaluate.

$$\begin{aligned} y''(t) + 2y'(t) + 5y(t) &= e^{-t}(-11\sin(2t) + 2\cos(2t)) + 2e^{-t}(3\sin(2t) + 4\cos(2t)) + 5e^{-t}\left(\sin(2t) - 2\cos(2t)\right) \\ &= e^{-t}(-11\sin(2t) + 2\cos(2t) + 6\sin(2t) + 8\cos(2t) + 5\sin(2t) - 10\cos(2t)) \\ &= 0 \end{aligned}$$

Yes, this solves the differential equation.

6. Derive a formula for $\frac{d^2}{dx^2} f(g(x))$ using the chain rule and product rule, and use this formula to calculate

$$\frac{d^2}{dx^2}\sin\left(x^4 + 5x^2 + 2\right).$$

SOLUTION: We already have

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

So we want to evaluate the derivative of this. Use the product rule and chain rule.

$$\frac{d}{dx} f'(g(x))g'(x) = (f''(g(x))g'(x))g'(x) + f'(g(x))g''(x) = f''(g(x))g'(x)^2 + f'(g(x))g''(x)$$

We have $f(g) = \sin(g), g(x) = x^4 + 5x^3 + 3$. So $f'(g) = \cos(g)$, $f''(g) = -\sin(g)$, $g'(x) = 4x^3 + 10x$, $g''(x) = 12x^2 + 10$. Using the formula we have

$$\frac{d^2}{dx^2}\sin\left(x^4 + 5x^2 + 2\right) = -\sin(x^4 + 5x^2 + 2)(4x^3 + 10x)^2 + \cos(x^4 + 5x^2 + 2)(12x^2 + 10)$$