Examples

**Example 2.1 Evaluate Limits**

\[
\lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} = \lim_{h \to 0} \frac{(5) \frac{1}{5+h} - \frac{1}{5+h}}{h} = \lim_{h \to 0} \frac{5}{5(5+h)} - \frac{5}{5(5+h)} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} \frac{-1}{5(5+h)} = -\frac{1}{25}
\]

\[
\lim_{x \to 4} \frac{3(x-4)\sqrt{x+5}}{3 - \sqrt{x+5}} = \lim_{x \to 4} \frac{3(x-4)\sqrt{x+5}(3 + \sqrt{x+5})}{3 - \sqrt{x+5}(3 + \sqrt{x+5})} = \lim_{x \to 4} \frac{3(x-4)\sqrt{x+5}(3 + \sqrt{x+5})}{9 - (x+5)} = \lim_{x \to 4} \frac{3(x-4)\sqrt{x+5}(3 + \sqrt{x+5})}{4 - x} = \lim_{x \to 4} \frac{-3(4-x)(\sqrt{x+5}(3 + \sqrt{x+5})}{4 - x} = -3\sqrt{4 + 5}(3 + \sqrt{4 + 5}) = -54
\]

**Example 2.2 Evaluate left and right-hand limits**

\[
\lim_{x \to 3^-} \frac{1}{x - 3} = -\infty,
\]

because as \(x \to 3\) from the left, \(x < 3\) so \(x - 3 < 0\), the function takes a negative value. However, since the denominator goes to zero, the fraction “goes to” \(-\infty\).

\[
\lim_{x \to 3^+} \frac{1}{x - 3} = -\infty,
\]

because as \(x \to 3\) from the right, \(x > 3\) so \(x - 3 > 0\), the function takes positive values.

**Example 2.3 Infinite limits**
Consider
\[ \lim_{x \to 4} \frac{x - 5}{(x - 4)^2}. \]

What we must notice is that the denominator is a squared number. It is never negative. The numerator is negative for \( x < 5 \). So when \( x \) is close to 4, the fraction takes a negative value (negative divided by a positive). And because the denominator tends to zero as \( x \) approaches 4, we can say that the limit is \( \infty \).

**Example 2.4** Find vertical asymptotes of the following functions:

1. \( f(\theta) = \tan(\frac{\pi \theta}{10}) \)
2. \( g(x) = \frac{1}{\sqrt{x} \sec(x)} \)
3. \( h(x) = e^{1/x} \)

1. \( \tan(x) \) is undefined if \( x = \frac{(2k+1)\pi}{2} \) for any integer \( k \). So \( f(\theta) \) will have a vertical asymptote if

\[ \frac{\pi \theta}{10} = \frac{(2k + 1)\pi}{2} \]

which is the same as \( \theta = 10k + 5 \), for any integer \( k \) (e.g. when \( \theta = 5, 15, -5, -15, \ldots \)).

2. Recall that \( \sec(x) = \frac{1}{\cos(x)} \). So we may re-write this as

\[ g(x) = \frac{\cos(x)}{\sqrt(x)}. \]

The function is not defined for \( x < 0 \) because it involves a \( \sqrt{x} \). The denominator = 0 only when \( x = 0 \), and because the numerator is defined for all \( x > 0 \) we only have a vertical asymptote when \( x = 0 \).

3. The function \( e^y \to \infty \) as \( y \to \infty \). So \( e^{1/x} \) goes to infinity as \( x \to 0^+ \) (because then \( \frac{1}{x} \to \infty \)). There will be asymptotic behavior from the right. From the left, however, as \( x \to 0^- \), \( \frac{1}{x} \to -\infty \) and \( e^{1/x} \to 0 \). So we do have a vertical asymptote at \( x = 0 \) but asymptotic behavior only on the right side of this asymptote.

**Example 2.5** Consider the function \( f(x) = x^{2/3} \). For \( x = h \), find the slope of the line between \((0,0)\) and \((h, f(h))\). Call this \( m(h) \). Determine \( \lim_{h \to 0} m(h) \).

First, by evaluating the function at \( x = h \) we get

\[ (h, f(h)) = (h, h^{2/3}). \]

The slope formula gives:

\[ \frac{y_2 - y_1}{x_2 - x_1} = \frac{h^{2/3} - 0}{h - 0} = \frac{h^{2/3}}{h} = h^{2/3} h^{-1} = h^{-1/3} = \frac{1}{\sqrt[3]{h}}. \]

By taking the right-hand and left-hand limits, we get:

\[ \lim_{h \to 0^+} \frac{1}{\sqrt[3]{h}} = +\infty, \]

since the function takes positive values for \( h > 0 \). Similarly we get

\[ \lim_{h \to 0^-} \frac{1}{\sqrt[3]{h}} = -\infty. \]

Because the right and left-hand limits do not agree, the limit does not exist.