Math 180: Calculus I **Fall 2014**

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Examples

Example 2.1 *Evaluate Limits*

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} &= \lim_{h\to 0} \frac{\frac{(5)}{(5)}\frac{1}{5+h} - \frac{1}{5}\frac{(5+h)}{(5+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{5}{5(5+h)} - \frac{5+h}{5(5+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{5(5+h)}}{h} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{5(5+h)} \frac{1}{\cancel{h}} \\
&= -\frac{1}{25}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} &= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \frac{(3+\sqrt{x+5})}{(3+\sqrt{x+5})} \\
&= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}(3+\sqrt{x+5})}{9-(x+5)} \\
&= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}(3+\sqrt{x+5})}{4-x} \\
&= \lim_{x\to 4} \frac{-3\cancel{(4-x)}\sqrt{x+5}(3+\sqrt{x+5})}{\cancel{4-x}} \\
&= -3\sqrt{4+5}(3+\sqrt{4+5}) \\
&= -54
\end{aligned}$$

Example 2.2 *Evaluate left and right-hand limits*

$$\lim_{x\to 3^-} \frac{1}{x-3} = -\infty,$$

because as $x \to 3$ from the left, $x < 3$ so $x - 3 < 0$, the function takes a negative value. However, since the denominator goes to zero, the fraction "goes to" $-\infty$.

$$\lim_{x\to 3^+} \frac{1}{x-3} = -\infty,$$

because as $x \to 3$ from the right, $x > 3$ so $x - 3 > 0$, the function takes positive values.

Example 2.3 *Infinite limits*

Consider

$$\lim_{x\to 4} \frac{x-5}{(x-4)^2}.$$

What we must notice is that the denominator is a squared number. It is never negative. The numerator is negative for $x < 5$. So when x is close to 4, the fraction takes a negative value (negative divided by a positive). And because the denominator tends to zero as x approaches 4, we can say that the limit is ∞.

Example 2.4 *Find vertical asymptotes of the following functions:*

1. $f(\theta) = \tan(\frac{\pi\theta}{10})$
2. $g(x) = \frac{1}{\sqrt{x}\sec(x)}$
3. $h(x) = e^{1/x}$

1. $\tan(x)$ is undefined if $x = \frac{(2k+1)\pi}{2}$ for any integer k. So $f(\theta)$ will have a vertical asymptote if

$$\frac{\pi\theta}{10} = \frac{(2k+1)\pi}{2}$$

which is the same as $\theta = 10k + 5$, for any integer k (e.g. when $\theta = 5, 15, -5, -15, ...$).

2. Recall that $\sec(x) = \frac{1}{\cos(x)}$. So we may re-write this as

$$g(x) = \frac{\cos(x)}{\sqrt{(x)}}.$$

The function is not defined for $x < 0$ because it involves a $\sqrt{x}$. The denominator $= 0$ only when $x = 0$, and because the numerator is defined for all $x > 0$ we only have a vertical asymptote when $x = 0$.

3. The function $e^y \to \infty$ as $y \to \infty$. So $e^{1/x}$ goes to infinity as $x \to 0^+$ (because then $\frac{1}{x} \to \infty$). There will be asymptotic behavior from the right. From the left, however, as $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $e^{1/x} \to 0$. So we do have a vertical asymptote at $x = 0$ but asymptotic behavior only on the right side of this asymptote.

Example 2.5 *Consider the function* $f(x) = x^{2/3}$. *For* $x = h$*, find the slope of the line between* $(0, 0)$ *and* $(h, f(h)$. *Call this* $m(h)$. *Determine* $\lim_{h\to 0} m(h)$.

First, by evaluating the function at $x = h$ we get

$$(h, f(h)) = (h, h^{2/3}).$$

The slope formula gives:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{h^{2/3} - 0}{h - 0} = \frac{h^{2/3}}{h} = h^{2/3}h^{-1} = h^{-1/3} = \frac{1}{\sqrt[3]{h}}.$$

By taking the right-hand and left-hand limits, we get:

$$\lim_{h\to 0^+} \frac{1}{\sqrt[3]{h}} = +\infty,$$

since the function takes positive values for $h > 0$. Similarly we get

$$\lim_{h\to 0^-} \frac{1}{\sqrt[3]{h}} = -\infty.$$

Because the right and left-hand limits do not agree, the limit does not exist.