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4.1 Counting Techniques

The Multiplication Rule: If you have two tasks to do, the first can be done in n_1 ways and the second can be done in n_2 ways, then there are $n_1 n_2$ ways to complete both tasks.

This can be illustrated for small problems nicely with a tree diagram, but this becomes impossible when the number of possibilities grows.

Generalized Multiplication Rule: If k tasks can each be done in n_1, n_2, \dots, n_k ways respectively, then there are $n_1 n_2 \cdots n_k$ ways to do all k tasks.

Definition 4.1. A **permutation** is an arrangement of all or part of a set of objects.

It is handy to use the **factorial** notation, $n! = n(n-1) \cdots (2)(1)$, for any integer $n > 0$. $0! = 1$ by definition.

Theorem 4.2. *The number of permutations of n objects is $n!$.*

This can be proved by induction on n .

Theorem 4.3. *The number of permutations of r out of n objects is*

$${}_n P_r = \frac{n!}{(n-r)!}.$$

This can be proved from the previous theorem. If we are only interested in the number of ways the first r items are ordered from the whole set of n items, we must realize that for any ordering of the first r items, there must be $(n-r)!$ ways for the remaining items to be ordered in the end of the list. So we divide by this factor to correct for repeated counts.

Theorem 4.4. *The number of permutations of n objects in a circle is $(n-1)!$.*

This is because any one of the n items may be oriented as the first object in the circle, so we divide $n!$ by n to correct for repeated counts.

Theorem 4.5. *The number of ways of partitioning n objects into r cells of sizes n_1, n_2, \dots, n_r where $n_1 + n_2 + \cdots + n_r = n$ is*

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

What matters is which objects go into each bin, not the order in which they go. So to correct for repeated counts like we did before, we divide by the number of ways each bin may be ordered ($n_i!$).

Definition 4.6. A **combination** is an un-ordered selection of r objects out of n , given the notation ${}_n C_r$ or $\binom{n}{r}$.

Theorem 4.7. *The number of combinations of r out of n objects is given by*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This can easily be seen as a corollary of the previous theorem where there are two cells - those objects selected and those not selected.

Exercise 4.8. Prove the following:

1. $\binom{n}{0} = \binom{n}{n} = 1$ for any positive integer n .
2. $\binom{n}{1} = \binom{n}{n-1} = n$ for any positive integer n .
3. $\binom{n}{r} = \binom{n}{n-r}$ for any positive integers $r \leq n$.

Exercise 4.9. Justify the Binomial Theorem:

$$(x + y)^n = \sum_{r=0}^n \binom{n}{r} x^r y^{n-r}$$

Exercise 4.10. Using the Binomial Theorem, how many subsets are there for a set of cardinality n ?

Exercise 4.11. Prove that

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}.$$

4.1.1 Commands in the TI-84