Example) $X \sim Gamma(\overset{\alpha}{5}, \overset{\beta}{10})$. Find $P(X < 60)$.

$$\int_0^{60} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx$$

substitute
$y = x/\beta = x/10$
$dy = \frac{dx}{\beta}$

$$= \int_0^{6} \frac{1}{\Gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-y} \frac{dx}{\beta}$$

$$= \int_0^{6} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy = F(6; 5) \approx .7149$$

Example) The number of customers per hour at a coffee shop follows a Poisson distribution, with 90 customers on average per hour. Let Y be the number of minutes until 10 customers come in. What distribution does Y follow?

$\lambda = 90$

$X = \#$ in 1 hour $\quad X \sim Poisson(90)$

$Y =$ time until 10 $\quad Y \sim Gamma\left(10, \frac{1}{90}\right) \quad \beta = \frac{1}{\lambda}$

$$\int_0^{x} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy = 1 - poissoncdf(x, \alpha - 1)$$

What is the probability it takes fewer than 5 minutes for 10 customers to arrive?

$$P\left(Y < \tfrac{1}{12}\right) = \int_0^{\frac{1}{12}} \frac{1}{\left(\frac{1}{90}\right)^{10} \Gamma(10)} y^{10-1} e^{-\frac{y}{1/90}}\, dy$$

Let $u = \frac{y}{\frac{1}{90}}$
$du = \frac{dy}{\frac{1}{90}}$
$u = 90y$

$$= \int_0^{7.5} \frac{1}{\Gamma(10)} u^9 e^{-u}\, du$$

$$= \cancel{F(450,} = 1 - poissoncdf(7.5, 9) \approx .2236$$

What is the average amount of time for 10 customers to arrive?

$$E(Y) = \alpha \cdot \beta = 10 \cdot \frac{1}{90} = \frac{1}{9} \text{ hours}$$

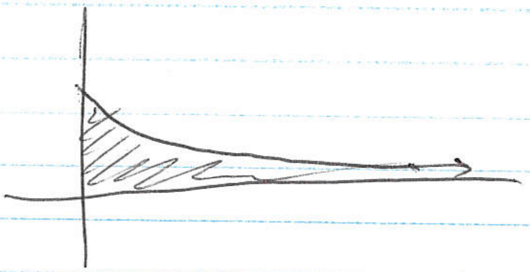
≈ 6.67 min.

Exponential Distribution

If $X \sim$ Gamma with $\alpha = 1$, we say
$X \sim$ Exponential(β)

X measures the length of time between occurrences of an event under a poisson process with $\lambda = \frac{1}{\beta}$

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \Gamma(1) = 0! = 1$$



$$\int_0^\infty \frac{1}{\beta} e^{-x/\beta}\, dx$$

$$= \frac{1}{\cancel{\beta}} (-\cancel{\beta})\, e^{-x/\beta} \Big]_0^\infty$$

$$= 0 - - e^{-0/\beta} = 1$$

$$F(x) = \int_0^x \frac{1}{\beta} e^{-t/\beta}\, dt = 1 - e^{-t/\beta}$$

$X \sim Exp(5)$

$$P(X < 3) = F(3) = 1 - e^{-3/5}$$

$$= .4512$$

Chi-Squared Distribution

$X \sim \text{Gamma} \quad \alpha = \frac{\nu}{2}, \beta = 2$

$X \sim \chi^2(\nu)$ with ν degrees of freedom

$$f(x) = \begin{cases} \frac{1}{(2)^{\nu/2}\Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} e^{-x/2} \\ 0 & x < 0 \end{cases}$$

If $Z \sim N(0,1)$, $Z^2 \sim \chi^2(1)$

$Y = Z^2 \qquad P(Y<0) = 0$

For $y \geq 0$ $\quad P(Y<y) = P(|Z| < \sqrt{y})$

$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

$$= 2\Phi(\sqrt{y}) - 1$$

$$f_Y(y) = 2\frac{d}{dy}\Phi(\sqrt{y}) - 0$$

$$= 2\frac{d}{dy}\left[\int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right]$$

$$= 2\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} (\sqrt{y})'$$

$$= \cancel{2}\frac{1}{\sqrt{2}\sqrt{\pi}} e^{-\frac{y}{2}} \left(\frac{1}{\cancel{2}} y^{-\frac{1}{2}}\right) \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} e^{-\frac{y}{2}} y^{-\frac{1}{2}} \qquad \alpha = \frac{1}{2} \ \beta = 2$$