Final Exam Review

STAT 381, Spr15

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1 Solutions To Review Problems from Ch 1-6

1. Find the mean, median, mode, range, min, max, Q_1, Q_3, IQR, variance and standard deviation for the following sample data (by hand or by calculator):

   $$3, 5, 5, 6, 7, 8, 10, 12, 12.5, 15, 15, 15, 16$$

   Mean: 9.9615, median: 10, Mode: 15, range: 16-3=13, min=3, max=16, Q_1=5.5, Q_3=15, IQR=15-5.5=9.5, $s^2 = 20.76923$, $s = 4.55732$

2. An electrical current has to pass through 3 checkpoints: at the first there are 4 gates, at the second there are 2 gates, and at the third there are 5 gates. How many paths are there through the circuit?
   $(4)(2)(5) = 40$ paths.

3. A friend is letting you borrow 3 movies, and he has 23 to choose from. How many selections are possible?
   $\binom{23}{3} = 1771$

4. Your extended family is cramming into a bus. There are 50 seats on the bus and your family has 37 members. How many ways can the 37 of you be seated on the bus (take note of the empty seats)?
   First 37 out of the 50 seats will have people in them, then the 37 people need to be ordered in those seats, so there are $\binom{50}{37}37! = P(50, 37)$ ways.

5. Of the 27 students in class, 13 of them have studied. If I randomly call 3 names, what is the probability that at least one of them studied? If I call on 3 names, what is the probability that all 3 have studied given that at least one of them has studied?

   $$P(\text{at least 1 studied}) = 1 - P(\text{0 studied}) = 1 - \frac{\binom{14}{3}}{\binom{27}{3}} = 1 - .12444 = .87556.$$

   $$\begin{aligned} P(\text{ 3 studied}|\text{at least 1 studied}) &= \frac{P(\text{3 studied} \cap \text{at least 1 studied})}{P(\text{at least 1 studied})} \\ &= \frac{P(\text{3 studied})}{.87556} \\ &= \frac{\binom{13}{3}/\binom{27}{3}}{.87556} \\ &= \frac{.09778}{.87556} \\ &= .11168 \end{aligned}$$

6. Your aunt told you that there was a 30% chance she would vacation in Rome, otherwise she'd go to Madrid. There's a 10% chance she'd be robbed in Rome and a 20% chance she'd be robbed in Madrid. If you find out she got robbed, what is the probability it happened in Rome?
Use Bayes' Theorem

$$\begin{aligned} P(Rome|Robbed) &= \frac{P(Rome)P(Robbed|Rome)}{P(Rome)P(Robbed|Rome) + P(Madrid)P(Robbed|Madrid)} \\ &= \frac{(.3)(.1)}{(.3)(.1) + (.7)(.2)} \\ &= .17647 \end{aligned}$$

7. $P(A) = .5, P(B) = .4$ and the two events are independent. What is $P(A \cup B')$?

$$\begin{aligned} P(A \cup B') &= P(A) + P(B') - P(A \cap B') && \text{inclusion exclusion} \\ &= .5 + (1 - P(B)) - P(A \cap B') && \text{complement rule} \\ &= .5 + (1 - .4) - P(A)P(B') && \text{independence} \\ &= .5 + .6 - (.5)(.6) \\ &= .8 \end{aligned}$$

8. Find the value of a which makes the following a valid pmf. Then find its cdf, $P(X > 4), E(X), E(X + 4), Var(X), E(ln(X))$.

x	1	2	3	4	5
$P(x)$	.21	.04	.12	a	.33

Since $.21 + .04 + .12 + a + .33 = 1$, $a = .30$. The cdf is

$$F(x) = \begin{cases} 0 & x < 1 \\ .21 & 1 \leq x < 2 \\ .25 & 2 \leq x < 3 \\ .37 & 3 \leq x < 4 \\ .67 & 4 \leq x < 5 \\ 1 & 5 \leq x \end{cases}$$

$P(X > 4) = .33$, $E(X) = 3.5$, $E(X + 4) = E(X) + 4 = 7.5$, $Var(X) = 2.25$,
$E(ln(X)) = (.21)ln1 + (.04)ln2 + (.12)ln3 + (.3)ln4 + (.33)ln5 = 1.10656$

9. Find the value of c which makes the following a valid pdf for random variable X. Then find its cdf, $P(X > 4), E(X), E(X + 4), Var(X)$

$$f(x) = \begin{cases} cx^{1/3} & 3 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_3^6 cx^{1/3}dx = \frac{3c}{4}x^{4/3}\Big|_3^6 = \frac{3c}{4}(6^{4/3} - 3^{4/3}) = 4.932c$$

So $c \approx .2028$
The cdf is

$$\int_3^x .2028x^{1/3}dt = \frac{3}{4}(.2028)x^{4/3}\Big|_3^x = .1521(x^4/3 - 3^4/3) = .1521x^{4/3} - .658098$$

So

$$F(x) = \begin{cases} 0 & x < 3 \\ .1521x^{4/3} - .658098 & 3 \le x < 6 \\ 1 & 6 \le x \end{cases}$$

$P(X > 4) = 1 - P(X \le 4) = 1 - F(4) = 1 - (.1521(4)^{4/3} - .658098) = 1 - .3077 = .6923$

$$E(X) = \int_3^6 x.2028x^{1/3}dx = \int_3^6 .2028x^{4/3}dx = \frac{3}{7}.2028x^{7/3}\Big|_3^6 = 4.557$$

$$E(X + 4) = E(X) + 4 = 8.557$$

$$E(X^2) = \int_3^6 x^2.2028x^{1/3}dx = \int_3^6 .2028x^{7/3}dx = \frac{3}{10}.2028x^{10/3}\Big|_3^6 = 21.5104$$

$$Var(X) = E(X^2) - E(X)^2 = 21.5104 - 4.557^2 = .74053$$

10. For independent random variables X and Y, $E(X) = 4, Var(X) = 13, E(Y) = 2, Var(Y) = 10$. Find $E(X - Y), Var(X - Y), E(2X + 3Y), Var(2X + 3Y)$
$E(X - Y) = E(X) - E(Y) = 4 - 2 = 2$
$Var(X - Y) = Var(X) + (-1)^2Var(Y) = Var(X) + Var(Y) = 13 + 10 = 23$
$E(2X + 3Y) = 2E(X) + 3E(Y) = 2(4) + 3(2) = 14$
$Var(2X + 3Y) = 2^2Var(X) + 3^2Var(Y) = 4(13) + 9(10) = 142$

11. If 35% of monarch butterfly caterpillars have spots, and a random sample of 16 are captured, what is the probability of getting at least 5 with spots? What is the expected number of caterpillars with spots, and its standard deviation? If you are going to keep capturing caterpillars until you get one with spots, what is the probability you will need *at least* 3 caterpillars? What is the expected number of caterpillars you'll need to catch, and what is the standard deviation?
X (the number of caterpillars with spots in a sample of 16) follows $Binom(16, .35)$ distribution.
$P(X \ge 5) = 1 - P(X \le 4) = 1 -$ `binomcdf(15,.35,4)` $= .7108$.
$E(X) = np = 16(.35) = 5.6$, $SD(X) = \sqrt{Var(X)} = \sqrt{npq} = \sqrt{16(.35)(.65)} = 1.9079$
Y (the number of caterpillars you catch until the first one with spots) follows a $Geom(.35)$ distribution.
$P(Y \ge 3) = 1 - P(X \le 2) = 1 -$ `geometcdf(.35,2)` $= .4225$
$E(Y) = 1/p = 2.85714, SD(Y) = \sqrt{Var(Y)} = \sqrt{q/p^2} = 2.3035$

12. If the average number of car alarms during 4 hours is 1 in your neighborhood, and it is a Poisson process, what is the probability that you can get 8 hours of sleep without a car alarm going off? What is the expected number of car alarms during 24 hours?
If the number of car alarms in 4 hours is a Poisson random variable with $\lambda = 1$, then X,the number of car alarms in 8 hours, is a Poisson random variable with $\lambda = 2$.
$P(X = 0) =$ `poissonpdf(2,0)` $= .1353$ is the probability of 8 hours with no car alarm.
The number of alarms in 24 hours is a Poisson r.v. with $\lambda = 6$, so the average number of car alarms in 24 hours is 6.

13. If you arrive at work between 8:45 and 9:25 with uniform likelihood, what is the probability you will get in before 9am? What is the probability that happens at least 3 out of 5 days in a week (assuming different days' arrival times are independent)? What is your average arrival time, and what is the standard deviation?
Let X be the hour you arrive. 8:45 we'll write as -15 and 9:25 we'll write as 25, so we'll use 9:00am as the 0.
$P(X < 9am) = P(-15 < X < 0) = (0 - -15)/(25 - -15) = .375$
If Y is the number of days in a week you arrive before 9am, Y follows $Binom(5, .375)$
$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 -$ `binomcdf(5,.375,2)` $= .2752$
$E(X) = (25 - 15)/2 = 5$ which is 9:05am.
$Var(X) = \frac{1}{12}(25 - -15)^2 = 133.333$, so $SD(X) = \sqrt{133.333} = 11.547$ minutes.

14. If the cost of gas at a Chicago gas station is approximately normally distributed with a mean price of \$2.789 and a standard deviation of \$0.15, what is the probability that the next gas station you see will have a price above \$3.00 per gallon?
$P(X > 3.00) =$ `normalcdf(3,10000,2.789,.15)` $= .0798$

15. You're busily working through a test. If the length of time to complete each question follows an exponential distribution with mean 1.3 minutes, what is the probability that a single problem takes you more than 4 minutes to complete? what is the expected number of problems you could complete in 50 minutes?
Letting X be the random amount of time a question takes,
$$P(X > 4) = \int_4^\infty \frac{1}{1.3} e^{-x/1.3} dx = -e^{-x/1.3}\Big|_4^\infty = e^{4/1.3} = .0461$$
The number of problems you can complete in 1 minute is on average then 1/1.3, so in 50 minutes you would expect to complete 50/1.3=38.46 problems.

2 Review Problems from Ch 8-10

1. If the probability that a random voter approves the governor is .45:
   (a) Suppose a random sampling of 60 voters is polled. Let X be the precise number of voters who support the governor. What distribution does X follow?
   X follows a Binomial distribution with $n = 60, p = .45$

(b) The Central Limit Theorem may be used to approximate the distribution of X by what distribution?
X should be approximately normal with $\mu = np = 27$ and $\sigma = \sqrt{npq} = 3.854$

2. A sampling of 15 lab grown diamonds has a mean clarity of 6.43 with standard deviation .22.

   (a) Give a 95% confidence interval for the mean clarity of this lab procedure's diamonds.
   A small sample with unknown population standard deviation requires a 1-sample T interval. It is

$$\bar{x} \pm t_{\alpha/2} s/\sqrt{n} = 6.43 \pm 2.145(.22/\sqrt{15}) = (6.3082, 6.5518)$$

   (b) Test with 5% significance whether the mean clarity is equal to 6.5 or not. Give the p-value.
   $H_0 : \mu = 6.5$
   $H_1 : \mu \neq 6.5$
   $t = (6.43 - 6.5)/(.22/\sqrt{15}) = -.1.2323$
   The P-value is $2P(T > 1.2323) = .2382$, which is much greater than the significance level of .05, so we would not reject the null hypothesis.

   (c) Test with 5% significance whether the mean clarity is at least 6.5 or not. Give the p-value.
   $H_0 : \mu = 6.5$
   $H_1 : \mu < 6.5$
   As before, $t = -1.2323$.
   The P-value for this lower-tailed test is $P(T < -1.2323) = .1191$, again this is large enough that we would still not want to reject the null hypothesis.