Problem 1 (Munkres §41; #3).
(a) Show that the kernel of \( \mathbb{Z}/n \rightarrow \mathbb{Z}/n \) is generated by \( \{n/d\} \), where \( d = (m, n) \).
(b) Show that a quotient of a cyclic group is cyclic
(c) Prove Lemma 41.4: There is an exact sequence
\[
0 \rightarrow \mathbb{Z}/d \rightarrow \mathbb{Z}/n \xrightarrow{m} \mathbb{Z}/n \rightarrow \mathbb{Z}/d \rightarrow 0
\]
where \( d = \gcd(m, n) \).

(a) Let \( d = (m, n) \). Then notice that \( d \cdot \frac{n}{d} = n \equiv 0 \in \mathbb{Z}/n \), so \( \frac{n}{d} \) is in the kernel of the map \( \cdot m \). Now let \( k \in \mathbb{Z}/n \) satisfy \( mk = 0 \in \mathbb{Z}/n \). Then we have \( mk = c'n \) for some \( c' \in \mathbb{Z} \). We can write \( n = d \cdot \frac{n}{d} \), and so we have \( mk = c'd\frac{n}{d} \). Dividing both sides by \( d \), we find \( \frac{d}{n}k = c'\frac{n}{d} \) where \( c' \frac{n}{d} \in \mathbb{Z} \). But \( \frac{n}{d} \) is relatively prime to \( n \), and so letting \( k = (\frac{n}{d})^{-1} c' \frac{n}{d} \), we see that \( k \in \{n/d\} \) and we are done.

(b) Let \( G \) be a cyclic group, and let \( H < G \) be a subgroup. Note that \( G \) is abelian, so that \( H \) is normal and hence we can consider \( G/H \) as a group. Let \( G = \langle g \rangle \), and let \( \overline{h} \in G/H \), where \( h \in G \) is a representative of \( \overline{h} \). We can write \( h = nh \). If \( n > 0 \), then we have \( h = g + g + \cdots + g \), so that \( \overline{h} = \overline{g} + \overline{g} + \cdots + \overline{g} \), i.e. \( \overline{h} = n\overline{g} \). If \( n \leq 0 \), we have similarly \( h = -g - g - \cdots - g \), so that \( \overline{h} = -\overline{g} - \overline{g} - \cdots - \overline{g} \), and hence \( \overline{h} = n\overline{g} \). Thus \( G/H = \langle \overline{g} \rangle \).

This could also be proven along these lines, which is how I first did it, but the above is a nicer proof:
Let \( G \) be a cyclic group, generated by \( \{x\} \), and let \( H \) be a subgroup of \( G \). Note that cyclic groups are abelian, and hence \( H \) is normal. If \( H = 0 \), then \( G/H = G \) and hence is cyclic. So suppose \( H \neq 0 \). Then \( H \) has an element \( y = nx \) such that \( n > 0 \) is minimal. It is easy to see that \( H \) is generated by \( y \). Now let \( z \in G/H \) for some choice of representative \( z \in G \). Then \( z = m x \) for some \( m \in \mathbb{Z} \). But then we can find a unique \( c \in \mathbb{Z} \) such that \( mx + cnx = m'x \) where \( 0 \leq m' < n \). Since \( cnx = cy \in H \), we may assume that \( 0 \leq m < n \). Moreover since \( mx = m'x \in G/H \) iff \( mx - m'x = (m - m')x \in H \), this choice of representatives is unique since \( -n < m - m' < n \). Since this set of representatives is generated by \( x + H \), we have that \( G/H \) is cyclic.

(c) Let \( \mathbb{Z}/d \rightarrow \mathbb{Z}/n \) be given by sending \( k \mapsto k\frac{n}{d} \). This map is injective since \( n/d \) is an element of order \( d \) in \( \mathbb{Z}/n \). From part 1, we have that the kernel of the map \( \mathbb{Z}/n \rightarrow \mathbb{Z}/n \) is exactly the image of \( \varphi_1 \). The cardinality of \( \text{Im}(m) \) is equal to \( \frac{|\mathbb{Z}/n|}{|\ker(m)|} \). But \( |\mathbb{Z}/n| = n \), and \( |\ker(m)| = |\text{Im}(\varphi_1)| = d \) since the first part of the sequence is exact, and so the cardinality of \( \text{Im}(m) = \frac{n}{d} \). Hence the cardinality of the \( \mathbb{Z}/n \) modulo \( \text{Im}(m) \) is \( d \), and is cyclic since it is a quotient of a cyclic group. Hence taking \( \mathbb{Z}/n \rightarrow \mathbb{Z}/d \) by composing the quotient map with the map taking a generator of the quotient to 1, we have the desired exact sequence.

Problem 2 (Munkres §45, #3). Let \( X \) and \( Y \) be spaces such that \( H_0(X), H^n(X), H_0(Y), \) and \( H^n(Y) \) are infinite cyclic. Let \( f : X \rightarrow Y \) be a continuous map. Show that if
\[
f_* : H_0(X) \rightarrow H_0(Y)
\]
equals multiplication by \( d \), then (up to sign) so does
\[
f^* : H^n(Y) \rightarrow H^n(X).
\][Hint: Show \( \kappa \) is an isomorphism.]
Consider the Kronecker maps $\kappa_X : H^n(X;G) \to \text{Hom}(H_n(X), G)$ and $\kappa_Y : H^n(Y;G) \to \text{Hom}(H_n(Y), G)$. From class we know that we have the following commutative diagram
\[
\begin{array}{c}
H^n(Y;G) \xrightarrow{\kappa_Y} \text{Hom}(H_n(Y), G) \\
\downarrow f^* \quad \downarrow (f_*)^+
\end{array}
\quad \begin{array}{c}
H^n(X;G) \xrightarrow{\kappa_X} \text{Hom}(H_n(X), G)
\end{array}
\]
Let $G = \mathbb{Z}$. Since the chain complex is free, we know that $\mathbb{Z} \cong H^p(X;\mathbb{Z}) \xrightarrow{\kappa_X} \text{Hom}(H_p(X), \mathbb{Z}) \cong \mathbb{Z}$ is surjective. But a surjective map from $\mathbb{Z}$ to $\mathbb{Z}$ is an isomorphism, and so we have $\kappa_X$ is an isomorphism. Similarly, $\kappa_Y$ is an isomorphism. Since $\text{Hom}(H_p(X), \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ and $\text{Hom}(H_p(Y) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$, our commutative diagram becomes
\[
\begin{array}{c}
\mathbb{Z} \xrightarrow{\kappa_Y} \mathbb{Z} \\
\downarrow f^* \quad \downarrow (f_*)^+
\end{array}
\quad \begin{array}{c}
\mathbb{Z} \xrightarrow{\kappa_X} \mathbb{Z}
\end{array}
\]
Thus we have $f^* = \kappa^{-1}_X (f_*)^+$ $\kappa_Y$. But $((f_*)^+ \varphi)(x) = \varphi(f_*(x)) = \varphi(dx) = d \varphi(x)$ so $(f_*)^+$ is also multiplication by $d$. Since isomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are given by sending $1$ to $\pm 1$, we thus have $f^*(1) = \pm 1 \cdot d \pm 1 = \pm d$, and so by linearity, $f^*$ is multiplication by $\pm d$, depending on our choice of isomorphisms $\kappa_X, \kappa_Y$.

**Problem 3** (Munkres §47; #1). Consider the Kronecker maps $P^N$ as a CW complex consisting of a cell in each dimension $0 \leq k \leq n$ and the attaching map from $e_k$ sends $\partial e_k$ to $e_{k-1}(-1)^k$. Thus we have the following cellular chain complex
\[
0 \xrightarrow{d_{N+1} \equiv 0} \mathbb{Z} \xrightarrow{d_N = 1 + (-1)^N} \ldots \xrightarrow{d_3 \equiv 0} \mathbb{Z} \xrightarrow{d_2 \equiv 2} \mathbb{Z} \xrightarrow{d_1 \equiv 0} \mathbb{Z} \xrightarrow{d_0 \equiv 0} 0.
\]
To find the cohomology with coefficients in a group $G$, we find the following chain complex
\[
0 \xrightarrow{N \equiv 0} G \xrightarrow{d_{N-1} = 1 + (-1)^N} \ldots \xrightarrow{d_2 \equiv 0} G \xrightarrow{d_1 \equiv 2} G \xrightarrow{d_0 \equiv 0} G \xrightarrow{d_{-1} = 0} 0.
\]
If we let $G = \mathbb{Z}$, we have
\[
0 \xrightarrow{d_N \equiv 0} \mathbb{Z} \xrightarrow{d_{N-1} = 1 + (-1)^N} \ldots \xrightarrow{d_2 \equiv 0} \mathbb{Z} \xrightarrow{d_1 \equiv 2} \mathbb{Z} \xrightarrow{d_0 \equiv 0} \mathbb{Z} \xrightarrow{d_{-1} = 0} 0.
\]
so we end up with cohomology
\[
H^n(P^N; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & n = 0 \\
\mathbb{Z}/2 & n \text{ even and } 1 < n \leq N \\
\mathbb{Z} & n \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]
If we let $G = \mathbb{Z}/2$, we have
\[
0 \xrightarrow{d_N \equiv 0} \mathbb{Z}/2 \xrightarrow{d_{N-1} \equiv 0} \ldots \xrightarrow{d_2 \equiv 0} \mathbb{Z}/2 \xrightarrow{d_1 \equiv 2} \mathbb{Z}/2 \xrightarrow{d_0 \equiv 0} \mathbb{Z}/2 \xrightarrow{d_{-1} = 0} 0.
\]
so we end up with cohomology
\[
H^n(P^N; \mathbb{Z}/2) = \begin{cases} 
\mathbb{Z}/2 & 0 \leq n \leq N \\
0 & \text{otherwise}
\end{cases}
\]
If we let $G = \mathbb{Q}$, we have
\[
0 \xrightarrow{d_N \equiv 0} \mathbb{Q} \xrightarrow{d_{N-1} = 1 + (-1)^N} \ldots \xrightarrow{d_2 \equiv 0} \mathbb{Q} \xrightarrow{d_1 \equiv 2} \mathbb{Q} \xrightarrow{d_0 \equiv 0} \mathbb{Q} \xrightarrow{d_{-1} = 0} 0.
\]
so we end up with cohomology
\[
H^n(P^N; \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & n = 0 \\
0 & \text{otherwise}
\end{cases}
\]
Similarly for $\mathbb{P}^\infty$ which is obtained by having a 1-cell in every dimension and attaching by maps of degree $1 + (-1)^n$ in dimension $n$, we have
\[ H^n(\mathbb{P}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n \text{ even}, \ n > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ H^n(\mathbb{P}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2 \text{ for all } n \]

\[ H^n(\mathbb{P}^\infty; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0 \\ 0 & \text{otherwise} \end{cases} \]

**Problem 4** (Munkres §51; #2). Use the cellular chain complexes to compute the homology, with general coefficients \( G \), of \( T \# T \) and \( \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \) and \( \mathbb{P}^N \) and the \( k \)-fold dunce cap.

**\( T \# T \):** We can express \( T \# T \) as a chain complex consisting of 1 0-cell \( v \), 4 1-cells \( l_1, l_2, l_3, l_4 \), and 1 2-cell \( e \) attached by sending \( \partial e \) to \( l_1l_2^{-1}l_1^{-1}l_2^{-1}l_3l_4^{-1}l_4^{-1} \). Notice that \( d_1 = 0 \) since there is only 1 1-cell, and we also have \( d_2 = 0 \) since each 1-cell occurs twice, once with degree 1 and once with degree -1. Hence we have the following chain complex

\[ 0 \xrightarrow{d_3=0} \mathbb{Z} \xrightarrow{d_2=0} \mathbb{Z}^4 \xrightarrow{d_1=0} \mathbb{Z} \xrightarrow{d_0=0} 0 \]

To find the homology with coefficients in \( G \), we tensor the chain complex with \( G \) and find the following chain complex

\[ 0 \xrightarrow{d_3=0} G \xrightarrow{d_2=0} \mathcal{G} \xrightarrow{d_1=0} G \xrightarrow{d_0=0} 0 \]

since the 0 map remains the 0 map after tensoring. Hence we have the following homology

\[ H_n(X; G) = \begin{cases} G & n = 0, 2 \\ \mathcal{G} \oplus G \oplus G & n = 1 \\ 0 & \text{otherwise} \end{cases} \]

**\( \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \):** We can express \( \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \) as a chain complex consisting of 1 0-cell \( v \), 3 1-cells \( l_1, l_2, l_3 \), and 1 2-cell \( e \) attached by sending \( \partial e \) to \( l_1^2l_2^2 \). Notice that \( d_1 = 0 \) since there is only 1 1-cell. However, \( d_2(me) = 2m(l_1 + l_2 + l_3) \). Hence we have the following chain complex

\[ 0 \xrightarrow{d_3=0} \mathbb{Z} \xrightarrow{d_2=0} \mathbb{Z}^3 \xrightarrow{d_1=0} \mathbb{Z} \xrightarrow{d_0=0} 0 \]

Here we use the basis \( \{(1,0,0),(0,1,0),(1,1,1)\} \) for \( \mathbb{Z}^3 \). To find the homology with coefficients in \( G \), we tensor the chain complex with \( G \) and find the following chain complex

\[ 0 \xrightarrow{d_3=0} G \xrightarrow{d_2=0} \mathcal{G} \xrightarrow{d_1=0} G \xrightarrow{d_0=0} 0 \]

where \( d_2 \) is given by multiplication by 2 on the third \( G \) component. Thus we have the following homology

\[ H_n(X; G) = \begin{cases} G & n = 0 \\ \mathcal{G} \oplus G \oplus G/2G & n = 1 \\ \ker(G \xrightarrow{2} G) & n = 2 \\ 0 & \text{otherwise} \end{cases} \]

**\( \mathbb{P}^N \):** We can express \( \mathbb{P}^N \) as a cellular chain complex consisting of one cell in each dimension \( e_0, e_1, e_2, \ldots, e_N \) where the attaching map on \( e_k \) is given by sending \( \partial e_k \) to \( e_{k-1}e_{k-1}^{-1} \). Thus we have the following chain complex

\[ 0 \xrightarrow{d_{N+1}=0} \mathbb{Z} \xrightarrow{d_N=2} \cdots \xrightarrow{d_{2}=2} \mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z} \xrightarrow{d_0=0} 0 \]

Tensoring with \( G \), we find the following chain complex

\[ 0 \xrightarrow{d_{N+1}=0} G \xrightarrow{d_N=(1+(-1)^N)} \cdots \xrightarrow{d_{2}=2} G \xrightarrow{d_1=0} G \xrightarrow{d_0=0} 0 \]
Thus we have the following homology

$$H_n(X; G) = \begin{cases} 
G & n = 0 \\
G/2G & 1 \leq n < N, \ n \equiv 1 \pmod{2} \\
\ker(G \to \mathbb{Z}) & 1 \leq n < N, \ n \equiv 0 \pmod{2} \\
\ker(G \to \mathbb{Z}) & n = N \text{ is odd} \\
0 & n = N \text{ is even} \\
0 & \text{otherwise}
\end{cases}$$

**k-fold dunce cap:** We can express the $k$-fold dunce cap $X$ as a cellular chain complex consisting of one 0-cell $v$, one 1-cell $l$, and one 2-cell $e$ attached by sending $\partial e$ to $l^k$. We have $d_1 \equiv 0$, and $d_2(e) = kl$. Thus we have the following cellular chain complex

$$0 \xrightarrow{d_3 = 0} \mathbb{Z} \xrightarrow{d_2 = k} \mathbb{Z} \xrightarrow{d_1 = 0} \mathbb{Z} \xrightarrow{d_0 = 0} 0.$$ 

Tensoring with $G$, we find the following chain complex

$$0 \xrightarrow{d_3 = 0} G \xrightarrow{d_2 = k} G \xrightarrow{d_1 = 0} G \xrightarrow{d_0 = 0} 0.$$ 

so we have the following homology

$$H_n(X; G) = \begin{cases} 
G & n = 0 \\
G/kG & n = 1 \\
\ker(G \to \mathbb{Z}) & n = 2 \\
0 & \text{otherwise}
\end{cases}$$

**Problem 5** (Munkres §54; #1). Compute $A \otimes B$ and $A \ast B$ if

$$A = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6, \quad B = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/12.$$ 

Since $\mathbb{Z}/m \oplus \mathbb{Z}/n = \mathbb{Z}/(m,n)$, $\mathbb{Z} \oplus \mathbb{Z}/n = \mathbb{Z}/n$, and the tensor product commutes with direct sum, we obtain

$$A \otimes B = (\mathbb{Z} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbb{Z}/9) \oplus (\mathbb{Z} \otimes \mathbb{Z}/12)$$

$$\oplus (\mathbb{Z}/2 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/2 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/2 \otimes \mathbb{Z}/12)$$

$$\oplus (\mathbb{Z}/4 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/4 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/4 \otimes \mathbb{Z}/12)$$

$$\oplus (\mathbb{Z}/6 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}/9) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}/12)$$

$$\equiv \mathbb{Z} \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/6$$

$$\equiv \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/4)^3 \oplus (\mathbb{Z}/6)^3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/12$$

The computation for the torsion product is similar, excepting that $\mathbb{Z} \ast \mathbb{Z} = \mathbb{Z} \ast \mathbb{Z}/n = 0$, so we find

$$A \ast B = \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6$$

**Problem 6** (Hatcher p.205, #4). What happens if one defines homology groups $h_n(X; G)$ as the homology groups of the chain complex $\cdots \to \text{Hom}(G, C_{n+1}(X)) \to \text{Hom}(G, C_n(X)) \to \cdots$? More specifically, what are the groups $h_n(X; G)$ when $G = \mathbb{Z}$, $\mathbb{Z}/m$, and $\mathbb{Q}$?

If we define the homology groups $h_n(X; G)$ as the homology groups of the chain complex described, we are, in effect, seeing what part of the structure of $G$ can be preserved by homomorphism into $C_n(X)$, rather than seeing what aspects of the structure of $C_n(X)$ are preserved by tensoring with $G$. If we let $G = \mathbb{Z}$, then we have $\text{Hom}(\mathbb{Z}, C_n(X)) \simeq C_n(X)$ in an obvious way: $a \in C_n(X)$ is the image of the map taking $1$ to $a$. When we consider $G = \mathbb{Z}/n$, we have $\text{Hom}(\mathbb{Z}/n, C_n(X)) = 0$ since $C_n(X)$ is free, so that $h_n(X; \mathbb{Z}/n) = 0$ for all $n$. Similarly, $\text{Hom}(\mathbb{Q}, C_n(X)) = 0$ since we can divide any element of $\mathbb{Q}$ (and hence it’s image in $C_n(X)$, a free $\mathbb{Z}$-module) by any integer $N$, clearly an impossibility unless the map is the zero map, and so $h_n(X; \mathbb{Q}) = 0$ for all $n$.

To finish the calculation for $h_n(X, \mathbb{Z})$, let $\varphi \in \text{Hom}(\mathbb{Z}, C_n(X))$, and notice that we obtain an element of $\text{Hom}(\mathbb{Z}, C_{n-1}(X))$, $d_n(\varphi) = \varphi \circ d$. In this case, we note that for $a \in C_n(X)$, we have $d_n(\varphi a) = \varphi(da) = \varphi da$, so the map $d$ agrees with the normal $d$, and so $h_n(X; \mathbb{Z}) = H_n(X)$. 