An Introduction to Model Categories

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Examples of Model Categories $(\mathcal{C}, W)$

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- **Simplicial abelian groups** with weak equivalences $f : A \xrightarrow{\simeq} B$ if $H_*(NA) \xrightarrow{\cong} H_*(NB)$.

Definition of Model Categories

Definition: A *model category* is a category $\mathcal{C}$ with 3 classes of maps W, C, and F, satisfying 5 axioms as below.

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- acyclic cofibrations $A \overset{\simeq}{\hookrightarrow} B$
- acyclic fibrations $X \overset{\simeq}{\twoheadrightarrow} Y$

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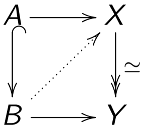
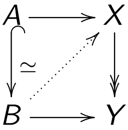
- *Lifting:* Lifts exist in the following squares:





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\begin{array}{ccc}
A & \longrightarrow & X \\
\big\downarrow & \nearrow & \big\downarrow \simeq \\
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$$

- *Factorization:* Any map $f : X \to Y$ factors in two ways

$$
X \xhookrightarrow{\simeq} Z \twoheadrightarrow Y \qquad\qquad X \hookrightarrow W \xrightarrow{\simeq} Y\ .
$$

Homotopy Category, Quillen Pair, Quillen Equivalence

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- Given $\mathcal{C}$, $\mathcal{D}$ model categories and an adjunction: $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, then (F, G) is a *Quillen pair* if F preserves cofibrations and G preserves fibrations. Then there is an induced adjunction:

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- If (LF, RG) induces an equivalence on the homotopy categories, then (F, G) is a *Quillen equivalence.*

$$\mathcal{C} \simeq_{\mathrm{QE}} \mathcal{D} \text{ and } \mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D}).$$

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- The *injective model structure* on ch^-:
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  C = monomorphisms in negative degree
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- Both extend to model structures on Ch:

$$\mathrm{Ch}_{\mathrm{Proj}} \simeq_{\mathrm{QE}} \mathrm{Ch}_{\mathrm{Inj}} \quad \text{and} \quad \mathrm{Ho}(\mathrm{Ch}_{\mathrm{Proj}}) \cong \mathrm{Ho}(\mathrm{Ch}_{\mathrm{Inj}})$$

Counter-examples and Examples

- There are examples of model categories $\mathcal{C}, \mathcal{D}$ with $\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D})$, but there is no Quillen pair inducing this equivalence. So,
$$\mathcal{C} \not\simeq_{\mathsf{QE}} \mathcal{D}.$$

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- (N, Γ) form a Quillen pair and a Quillen equivalence
$$s\mathcal{A}b \simeq_{\mathsf{QE}} \mathrm{ch}^+ \text{ and } \mathrm{Ho}(s\mathcal{A}b) \cong \mathrm{Ho}(\mathrm{ch}^+)$$

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- (Schwede-S. 2003) N induces a functor on simplicial rings, and is part of a Quillen equivalence,
$$s\,\mathrm{Rings} \simeq_{\mathsf{QE}} \mathrm{DGA}^+ \ \text{ and } \ \mathrm{Ho}(s\,\mathrm{Rings}) \cong \mathrm{Ho}(\mathrm{DGA}^+)$$