Rational SO(2) equivariant ring spectra

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The main result

Let $SO(2) = \mathbb{T}$, $\mathcal{A}(\mathbb{T})$ is an explicit abelian category, with injective dimension 1, and $d\mathcal{A}(\mathbb{T})$ is the associated category with differentials.

Theorem (BGKS, The classification of rational $\mathbb{T}$-spectra)

There is a (zig–zag) of symmetric monoidal Quillen equivalences between rational $\mathbb{T}$-equivariant spectra and $d\mathcal{A}(\mathbb{T})$.

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} d\mathcal{A}(\mathbb{T}) \quad \text{and} \quad \mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \underset{\triangle}{\simeq} D\mathcal{A}(\mathbb{T}).$$

Corollary (BGKS, Rings and modules)

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}\text{-}\textit{rings} \underset{QE}{\simeq} d\mathcal{A}(\mathbb{T})\text{-}\textit{rings}$$

$$E\text{-}\textit{mod in } \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} \Theta E\text{-}\textit{mod in } d\mathcal{A}(\mathbb{T})$$

Existing work

Original Outline for $G = \mathbb{T}$, without monoidal structure

$$\begin{array}{ccc} \text{Greenlees 1999} & & \text{S. 2002} \\ \mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \underset{\triangle}{\cong} D\mathcal{A}(\mathbb{T}) & \Longrightarrow & \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T}) \end{array}$$

Existing work

Revised Outline

Greenlees 1999
$\mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \cong D\mathcal{A}(\mathbb{T})$

$\rightsquigarrow$

Greenlees-S.
$\mathbb{T}^n\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T}^n)$

$\rightsquigarrow$ (downward)

BGKS, with monoidal structure
$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T})$

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$\mathbb{T}^n\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T}^n)$

$\Downarrow$

$\rightsquigarrow$ (downward)

BGKS, with monoidal structure
$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\mathrm{QE}}{\simeq} d\mathcal{A}(\mathbb{T})$

$\Longrightarrow$ $\mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \underset{\triangle}{\cong} D\mathcal{A}(\mathbb{T})$

Rigidity

Corollary (S. 2002)

The model category of rational $\mathbb{T}$-equivariant spectra is rigid:

$$\mathrm{Ho}(\mathscr{C}) \underset{\triangle}{\cong} \mathrm{Ho}(\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}) \quad \Longrightarrow \quad \mathscr{C} \underset{QE}{\simeq} \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}$$

Can compute *All* Toda brackets here (use injective dimension 1).

Model as differential graded modules over an *explicit* rational DGA .

Appendix (S. 2002)

Toda brackets are determined by the triangulated structure on the homotopy category.

Existing Work.

General aim

For each compact Lie group G, find a simple algebraic category $\mathcal{A}(G)$ which is symmetric monoidally Quillen equivalent to $G\,\mathrm{Sp}_{\mathbb{Q}}$.

Existing monoidal work

- G finite (Barnes, 2009; Kedziorek, 2014)
- $G = \mathbb{T} = SO(2)$ (BGKS, 2017)

Existing non-monoidal work

- $G = O(2)$ (Barnes, 2016)
- $G = SO(3)$ (Kedziorek, to appear, preprint 2016)
- $G = \mathbb{T}^n$ (Greenlees- S., preprint 2016)

Cohomology theories and equivariance

Let G be group, X a based topological space with G action and let F^* be a cohomology theory.

We need equivariant cohomology theories

- $F^*(X)$ has a G-action.
- Consider EG the contractible, universal free G-space, $EG/G = BG$.
- $E\mathbb{T} = S^\infty \subset \mathbb{C}^\infty$, $B\mathbb{T} = \mathbb{C}P^\infty$.

Examples

- The borel construction: $F^*(X \wedge_G EG_+)$.
- Equivariant K–theory.
- Equivariant cobordism.

Equivariant cohomology theories

For V a representation of G, define S^V as the one-point compactification of V. Note, if $V \cong \mathbb{R}^n$, then $S^V \cong S^n$.

Definition

A G-**equivariant cohomology theory** F_G^* consists of cohomology theories such that

$$(F_G^{V \oplus W})^*(S^W \wedge X) \cong (F_G^V)^*(X).$$

Here F_G^* as an $RO(G)$-graded cohomology theory.

Theorem (Equivariant Brown representability)

A G-equivariant cohomology theory F_G^* is represented by a G-spectrum F_G. That is $F_G^*(A) = [\Sigma^\infty A, F_G]_*^G$.

Equivariant spectra

Definition

For G a compact Lie group, a G-**spectrum** X is a collection of based G-spaces $X(V)$ for each finite dimensional real representation V of G, along with structure maps

$$X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

For each V there is an equivalence of categories

$$- \wedge S^V \colon \mathrm{Ho}(G\,\mathrm{Sp}) \xrightarrow{\cong} \mathrm{Ho}(G\,\mathrm{Sp})$$

Example

For a G-space A, let $\Sigma^\infty A$ be the spectrum with $(\Sigma^\infty A)(V) = A \wedge S^V$.

Rational equivariant spectra

Definition

The **model category of rational** G**-spectra** $G\,\mathrm{Sp}_{\mathbb{Q}}$ is the category $G\,\mathrm{Sp}$ with weak equivalences those maps f such that $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all closed subgroups H of G.

Theorem (Rational equivariant Brown representability)

A rational G-equivariant cohomology theory F_G^* is represented by a rational G-spectrum F_G.

Existing Work.

General aim

For each compact Lie group G, find a simple algebraic category $\mathcal{A}(G)$ which is symmetric monoidally Quillen equivalent to $G\,\mathrm{Sp}_{\mathbb{Q}}$.

Existing monoidal work

- G finite (Barnes, 2009; Kedziorek, 2014)
- $G = \mathbb{T} = SO(2)$ (BGKS, 2017)

Finite G

Let $W_G H = N_G H/H$.

Theorem

Let G be a finite group. The category of rational G-spectra is symmetric monoidally Quillen equivalent to the product category

$$G\,\mathrm{Sp}_{\mathbb{Q}} \quad \underset{QE}{\simeq} \quad \prod_{(H)\leqslant G} \mathrm{Ch}_{\mathbb{Q}}[W_G H]$$

Greenlees-May 1992:	homotopy level equivalence
Schwede-S. 2003:	Quillen equivalence
Barnes 2009 and Kedziorek 2014:	symmetric monoidal equivalence

The finite case from Barnes 2009.

Facts for finite G

- $[\mathbb{S}, \mathbb{S}]_*^{G\mathbb{Q}} \cong A(G) \otimes \mathbb{Q} \cong \prod_{(H) \leqslant G} \mathbb{Q}$.
- The homotopy category is generated by $e_H \Sigma^\infty G/H_+$ for varying H.

Proposition (Barnes 2009)

There is a symmetric monoidal Quillen equivalence

$$G\,\mathrm{Sp}_{\mathbb{Q}} \underset{\prod}{\overset{\Delta}{\rightleftarrows}} \prod_{(H)\leqslant G} L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$$

$L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$ is the model category with weak equivalences those f with $e_H \pi_*^K(f) \otimes \mathbb{Q}$ an isomorphism for all $K \leqslant G$.

The finite case from Barnes 2009.

Let $W_G H = N_G H/H$

The model category $L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}}$ is generated by $e_H \Sigma^\infty G/H_+$. The homotopy of the self maps of this generator are concentrated in degree zero:

$$\pi_* F(e_H G/H_+, e_H G/H_+)^G \simeq W_G H_+$$

Proposition (Barnes 2009)

There is a Morita-type Quillen equivalence:

$$L_{e_H \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} F(e_H G/H_+, e_H G/H_+)^G\text{–mod}\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} \mathrm{Sp}_{\mathbb{Q}}[W_G H]$$

$$\mathrm{Sp}_{\mathbb{Q}}[W_G H] \underset{\mathrm{QE}}{\simeq} \mathrm{Ch}_{\mathbb{Q}}[W_G H]$$

The finite case from Kedziorek 2014

Proposition (Kedziorek 2014)

Let G be a finite group and $W_G H = N_G H / H$. There are symmetric monoidal Quillen equivalences

$$L_{e_H^G \mathbb{S}} G\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} \mathrm{Sp}_{\mathbb{Q}}[W_G H] \underset{QE}{\simeq} \mathrm{Ch}_{\mathbb{Q}}[W_G H]$$

Theorem (Kedziorek 2014)

Let G be a finite group. The category of rational G-spectra is symmetric monoidally Quillen equivalent to the following product.

$$G\,\mathrm{Sp}_{\mathbb{Q}} \underset{QE}{\simeq} \prod_{(H) \leqslant G} \mathrm{Ch}_{\mathbb{Q}}[W_G H]$$

$\mathbb{T}$–spectra

Facts for $\mathbb{T}$

- The homotopy category is generated by $\mathbb{T}/H_+$ for varying H.
- $[\Sigma^\infty \mathbb{T}/H_+, \Sigma^\infty \mathbb{T}/K_+]_*^{\mathbb{T}\mathbb{Q}}$ is not concentrated in degree zero.
- $[\mathbb{S}, \mathbb{S}]_*^{\mathbb{T}\mathbb{Q}} \cong A(G) \otimes \mathbb{Q} \cong \mathbb{Q}$.

Next, decompose $\mathbb{S}$. Let $\mathcal{F}$ be the family of finite subgroups of $\mathbb{T}$. Consider the cofibre sequence:

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}$$

There is a pullback square of $\mathbb{T}$-spectra

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & DE\mathcal{F}_+ \\ \downarrow & & \downarrow \\ \widetilde{E}\mathcal{F} & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \end{array}$$

$\mathbb{T}$–spectra

A pullback of model categories:

$$\begin{array}{ccc} S^{\bullet}\text{–mod} & \longrightarrow & DE\mathcal{F}_{+}\text{–mod} \\ \downarrow & & \downarrow \\ L_{\widetilde{E}\mathcal{F}}\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \longrightarrow & L_{DE\mathcal{F}_{+}\wedge\widetilde{E}\mathcal{F}}DE\mathcal{F}_{+}\text{–mod} \end{array}$$

Definition

$S^{\bullet}$–mod has **objects** $(X \xrightarrow{f} Y \xleftarrow{g} Z)$,
$f{:}X \wedge DE\mathcal{F}_{+} \to Y$ and $g{:}Z \to Y$ maps in $DE\mathcal{F}_{+}$–mod.

Example

For $X \in \mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}}$, $(X \wedge \widetilde{E}\mathcal{F} \to X \wedge \widetilde{E}\mathcal{F} \wedge DE\mathcal{F}_{+} \leftarrow X \wedge DE\mathcal{F}_{+})$ is in $S^{\bullet}$–mod.

The decomposition

There is a Quillen adjunction, not a Quillen equivalence.

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\text{pullback}}{\overset{S^{\bullet}\wedge -}{\rightleftarrows}} S^{\bullet}\text{-mod}$$

Derived unit map:

$$X \to \mathrm{pullback}(X \wedge \widetilde{E}\mathcal{F} \to X \wedge \widetilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \leftarrow X \wedge DE\mathcal{F}_+).$$

Since it is a weak equivalence for $\mathbb{S}$ and $X \wedge -$ commutes with pullbacks, it is a weak equivalence for all X.

The decomposition

There is a Quillen adjunction, not a Quillen equivalence.

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{\text{pullback}}{\overset{S^\bullet \wedge -}{\rightleftarrows}} S^\bullet\text{-mod}$$

For all X,

$$X \xrightarrow{\simeq} \text{pullback}(X \wedge \widetilde{E}\mathcal{F} \to X \wedge \widetilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \leftarrow X \wedge DE\mathcal{F}_+).$$

So, $S^\bullet \wedge -$ is full and faithful.

We make it essentially surjective by using cellularization.

Cellularization

Consider the "cells" $K = \{S^\bullet \wedge (\mathbb{T}/C_n)_+ \mid n \geqslant 1\} \cup \{S^\bullet \wedge (\mathbb{T}/\mathbb{T})_+\}$.

Definition

In K–cell–$S^\bullet$–mod, a map $f\colon M \to N$ is a weak equivalence if for all $k \in K$

$$[k, M]^{S^\bullet} \xrightarrow{\cong} [k, N]^{S^\bullet}$$

Replacing Ho($S^\bullet$–mod) by the full subcategory generated by the images of K.

Theorem

There is a Quillen equivalence **(STEP 1)**

$$\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} \underset{pullback}{\overset{S^\bullet\wedge -}{\rightleftarrows}} K\text{–cell–}S^\bullet\text{–mod}$$

Fixed points

Lemma

Taking fixed points induces Quillen equivalences:

$$\begin{aligned} L_{\widetilde{E}\mathcal{F}}\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} &\simeq \mathrm{Sp}_{\mathbb{Q}} \\ DE\mathcal{F}_+\text{-mod} &\simeq (DE\mathcal{F}_+)^{\mathbb{T}}\text{-mod} \\ L_{DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F}}DE\mathcal{F}_+\text{-mod} &\simeq L_{(DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F})^{\mathbb{T}}}(DE\mathcal{F}_+)^{\mathbb{T}}\text{-mod} \end{aligned}$$

Define $S^{\bullet}_{top}$ using the right hand side of the above.

Theorem

There is a Quillen equivalence

$$S^{\bullet}\text{-mod} \underset{(-)^{\mathbb{T}}}{\overset{\longleftarrow}{\longrightarrow}} S^{\bullet}_{top}\text{-mod}$$

Fixed points

Lemma

Taking fixed points induces Quillen equivalences:

$$\begin{array}{rcl} L_{\widetilde{E}\mathcal{F}}\mathbb{T}\,\mathrm{Sp}_{\mathbb{Q}} & \simeq & \mathrm{Sp}_{\mathbb{Q}} \\ DE\mathcal{F}_+\text{-mod} & \simeq & (DE\mathcal{F}_+)^{\mathbb{T}}\text{-mod} \\ L_{DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F}}DE\mathcal{F}_+\text{-mod} & \simeq & L_{(DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F})^{\mathbb{T}}}(DE\mathcal{F}_+)^{\mathbb{T}}\text{-mod} \end{array}$$

Define $S^{\bullet}_{top}$ using the right hand side of the above.

Theorem

There is a Quillen equivalence by the cellularization principle [Greenlees and S. 2013] **(STEP 2)**

$$K\text{-cell-}S^{\bullet}\text{-mod} \underset{(-)^{\mathbb{T}}}{\overset{\longleftarrow}{\longrightarrow}} K^{\mathbb{T}}\text{-cell-}S^{\bullet}_{top}\text{-mod}$$

Algebraicization

Translate to DGAs

Theorem

There is a diagram of model categories of differential graded modules

$$S_t^\bullet = \left(\Theta DE\mathcal{F}_+^{\mathbb{T}}\text{-mod} \to L_A(\Theta DE\mathcal{F}_+^{\mathbb{T}})\text{-mod} \leftarrow \mathrm{Ch}_{\mathbb{Q}}\right)$$

such that the model categories below are symmetric monoidally Quillen equivalent. **(STEP 3)**

$$K^{\mathbb{T}}\text{-cell-}S_{top}^\bullet\text{-mod} \simeq K_t^{\mathbb{T}}\text{-cell-}S_t^\bullet\text{-mod}$$

We have isomorphisms of *commutative* rings

$$H_*(\Theta DE\mathcal{F}_+^{\mathbb{T}}) \cong \pi_*^{\mathbb{T}}(DE\mathcal{F}_+) \cong \mathcal{O}_{\mathcal{F}} = \prod_{n\geqslant 1} \mathbb{Q}[c_n]$$

hence $\Theta DE\mathcal{F}_+^{\mathbb{T}} \simeq \mathcal{O}_{\mathcal{F}}$ by a formality argument.

Formality

This simplifies $S_t^{\bullet}$ –mod to:

$$\mathcal{O}_{\mathcal{F}}\text{–mod} \to L_A\mathcal{O}_{\mathcal{F}}\text{–mod} \leftarrow \mathrm{Ch}_{\mathbb{Q}}$$

with A a ring object such that

$$H_*(A) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \mathrm{colim}_n\, \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \ldots, c_n^{-1}].$$

$\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is formal and hence

$$L_A\mathcal{O}_{\mathcal{F}}\text{–mod} \simeq \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{–mod}$$

Name the images of $K_t^{\mathbb{T}}$ as K_a under these simplifications.

Removing cellularization

Theorem

(STEP 4)

$$K_t^{\mathbb{T}}\text{–cell–}S_t^{\bullet}\text{–mod} \simeq$$

$$K_a\text{–cell–}(\mathcal{O}_{\mathcal{F}}\text{–mod} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{–mod} \leftarrow \mathrm{Ch}_{\mathbb{Q}}) = d\hat{\mathcal{A}}$$

$d\hat{\mathcal{A}}$ has objects $(M \to N \leftarrow V)$ with maps

$$\mathcal{E}^{-1}M \to N \leftarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

Originally, (1999) $d\mathcal{A}(\mathbb{T})$ has objects morphisms of $\mathcal{O}_{\mathcal{F}}$-modules

$$\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

such that β is an isomorphism after inverting $\mathcal{E}$.
Include $d\mathcal{A}(\mathbb{T})$ into $d\hat{\mathcal{A}}$ by taking $N = \mathcal{E}^{-1}M$.

Completing the proof

Use formality to identify the cells. Then cellularization implies structure maps of $d\hat{\mathcal{A}}$ are homology isomorphisms.

Theorem

The model categories $d\hat{\mathcal{A}}$ and $d\mathcal{A}(\mathbb{T})$ are Quillen equivalent. **(STEP 5)**

Theorem (BGKS, The classification of rational $\mathbb{T}$-spectra)

There is a (zig–zag) of symmetric monoidal Quillen equivalences between rational $\mathbb{T}$-equivariant spectra and $d\mathcal{A}(\mathbb{T})$.

The algebraic model for $\mathbb{T} = S^1$

Let $\mathcal{O}_{\mathcal{F}} = \prod_{n \geqslant 1} \mathbb{Q}[c_n]$ and $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \operatorname{colim}_n \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \ldots, c_n^{-1}]$, with $\deg(c_n) = -2$.

Definition (The algebraic model $\mathcal{A}(\mathbb{T})$)

Let $\mathcal{A}(\mathbb{T})$ be the category whose **objects** are morphisms of $\mathcal{O}_{\mathcal{F}}$-modules of the form

$$\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

such that β is an isomorphism after inverting $\mathcal{E}$.

A **morphism** is a pair (θ, ϕ) which makes the following square commute

$$\begin{array}{ccc} M & \xrightarrow{\beta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V \\ \downarrow \theta & & \downarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} \phi \\ M' & \xrightarrow{\beta'} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V' \end{array}$$

Let $d\mathcal{A}(\mathbb{T})$ be the associated category with differentials.