Rigidity and algebraic models for rational equivariant stable homotopy theory

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Joint work with John Greenlees

Question: Given stable model categories $\mathcal{C}$ and $\mathcal{D}$, when can a triangulated equivalence between $\mathcal{H}o(\mathcal{C})$ and $\mathcal{H}o(\mathcal{D})$ be realized by an underlying equivalence of model categories?

$$\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{H}o(\mathcal{D}) \overset{?}{\Rightarrow} \mathcal{C} \simeq_{Q} \mathcal{D}$$

Rigidity Theorem.(Schwede '07) If the homotopy category of a stable model category $\mathcal{C}$ and the homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of spectra.

$\mathcal{H}o$(Spectra) is rigid.

Examples: (Dugger - S. '04, '09)
If A and B are derived equivalent rings, then the model categories of differential graded modules are Quillen equivalent. That is,

$\mathcal{D}(A) \simeq_{\triangle} \mathcal{D}(B) \Rightarrow$ d.g. A-Mod $\simeq_Q$ d.g. B-Mod.

Corollary: $\mathcal{D}(A)$ is rigid among all stable model categories. That is, given a (nice) stable model category $\mathcal{C}$ such that $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(A)$ then $\mathcal{C} \simeq_Q$ d.g. A-Mod.

If A and B are differential graded algebras (DGAs), then there are counter-examples to this.

Definition. A DGA is *formal* if it is quasi-isomorphic to its homology ring. That is, $A \cong H_*A$.

Definition. A DGA is *intrinsically formal* if it is determined up to quasi-isomorphism by its homology ring. That is, A is intrinsically formal if and only if $H_*A \cong H_*B \Rightarrow A \simeq B$

Lemma. If A is an intrinsically formal DGA and $\mathcal{D}(A) \simeq_\triangle \mathcal{D}(B)$, then d.g. A-Mod $\simeq_Q$ d.g. B-Mod.

Here though, $\mathcal{D}(A)$ is not rigid among all stable model categories.

Theorem. (Schwede-S. '03, S. '07) Any (nice) *rational* stable model category $\mathcal{C}$ with a (set of) compact generator(s) is Quillen equivalent to DG modules over B a DGA (or DG category.)

This provides an algebraic model for any rational stable homotopy theory. This model is not necessarily explicit or useful for calculations.

Theorem. If A is an intrinsically formal, *rational* DGA then $\mathcal{D}(A)$ is rigid among all stable model categories. That is, for any (nice) stable model category $\mathcal{C}$ if $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(A)$, then $\mathcal{C} \simeq_{Q} A$-mod.

This also extends to intrinsically formal DG categories.

Question: Given a (nice) rational stable model category $\mathcal{C}$ how can we recognize $\mathcal{H}o(\mathcal{C})$, or the associated $\mathcal{D}(B)$, as rigid?

Conjecture.(Greenlees) For any compact Lie group G there is an abelian category $\mathcal{A}(G)$ such that

$$\mathbb{Q}\text{ - }G\text{-spectra} \simeq_Q \mathcal{A}(G)$$

where $\mathcal{A}(G)$ has injective dimension equal to the rank of G.

Verified for finite groups, SO(2), O(2) (G.-May, G., S., Barnes)

Theorem 1.(G.-S., '11) The conjecture holds for G any torus.

The rest of this talk will develop the algebraic model for $G = S^1$. The outline is similar for any torus.

We work on the model category level to find a model where we can use intrinsic formality.

Definitions. Let $\mathfrak{F} = \{F\}$ be the family of finite subgroups of G. Define

$$(E\mathfrak{F})^H = \begin{cases} \text{pt} & H \text{ finite} \\ \emptyset & H \text{ not finite} \end{cases}$$

Define $\widetilde{E}\mathfrak{F}$ as the cofiber of the map $E\mathfrak{F}_+ \to S^0$.

Define $DE\mathfrak{F}_+ = \text{Hom}(E\mathfrak{F}_+, S^0)$.
$\pi_* DE\mathfrak{F}_+ = H^* E\mathfrak{F}_+$ (all implicitly rational)

Step 1 Separate the fixed points.

Proposition. For $G = SO(2)$ there is a homotopy pullback of G-equivariant commutative ring spectra.

$$\begin{array}{ccc} S^0 & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \end{array}$$

Main Outline: Show S^0-modules are modeled by diagrams of modules over the three other corners. Note the fixed points of these three corners have intrinsically formal homotopy. Then the algebraic model is modules over the diagram of homotopy rings of the fixed point spectra.

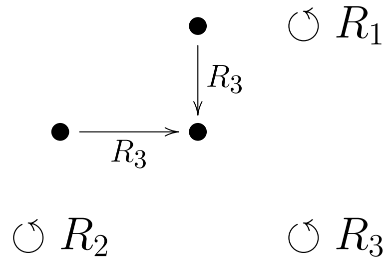
General Case: Assume given a homotopy pullback of rings (ring spectra or DGAs):

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_3 \end{array}$$

Let $R^{\lrcorner}$ denote the diagram of rings above with R deleted.

$$\begin{array}{ccc} & & R_1 \\ & & \downarrow \\ R_2 & \longrightarrow & R_3 \end{array}$$

Definition. $R^{\lrcorner}$-modules is the category of (d.g. or spectral) modules over the ring with three objects with $\mathrm{Hom}(1,3) = R_3$ and $\mathrm{Hom}(2,3) = R_3$.



Such a module is a collection $\{M_i\}_{i=1,2,3}$ of $\{R_i\}$-modules with structure maps $R_3 \otimes_{R_1} M_1 \to M_3$ and $R_3 \otimes_{R_2} M_2 \to M_3$. (The adjoints of these structure maps are an R_1-morphism $M_1 \to M_3$ and an R_2-morphism $M_2 \to M_3$.)

Note $R^{\lrcorner}$ determines such a module.

$R^{\lrcorner}$-Mod has three generators R-Mod has only one.

Proposition. The derived category of R-modules is equivalent to the localizing subcategory of $R^{\lrcorner}$-modules generated by $R^{\lrcorner}$. This equivalence is induced by a Quillen equivalence of model categories.

$$R\text{-Mod} \simeq_Q \text{cell}_{\{R^{\lrcorner}\}}\text{ - } R^{\lrcorner}\text{-Mod}$$

Proof. Consider the adjoint functors on the generators.

$$M \quad \to \quad R^{\lrcorner} \otimes_R M$$

$$\text{pullback}(\{M_i\}) \quad \leftarrow \quad \{M_i\}$$

Step 1:
Rational G-spectra are S^0-modules; apply above proposition with above square with $R^{\lrcorner} =$

$$\begin{array}{ccc} & & \widetilde{E}\mathfrak{F} \\ & & \downarrow \\ DE\mathfrak{F}_+ & \longrightarrow & DE\mathfrak{F}_+ \wedge \widetilde{E}\mathfrak{F}. \end{array}$$

Here the generators of S^0-modules are $\{G/H_+\}$, so we cellularize with respect to $\{G/H_+ \wedge R^{\lrcorner}\}_H$.

Conclude:

$$\mathbb{Q}\text{ - }G\text{-spectra} = S^0\text{-Mod} \simeq_1 \text{cell}_{\{G/H_+ \wedge R^{\lrcorner}\}}\text{ - }R^{\lrcorner}\text{-Mod}$$

Step 2: Move from G-spectra to spectra.

$$A\text{-Mod}_{(G\text{ - spectra})} \leftrightarrow A^G\text{-Mod}_{(\text{spectra.})}$$

Usually this is just an adjunction.

Here, this induces an equivalence on each of the cells $\{G/H_+ \wedge R^{\lrcorner}\}_H$ for each of the relevant rings.

$$S^0\text{-Mod}_G \simeq_1 \text{cell-}R^{\lrcorner}\text{-Mod}_G \simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod}$$

Step 3: Make algebraic:
rational commutative ring spectra are modeled by
rational commutative DGAs

$$\simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod} \simeq_3 \text{cell-d.g.-}(R^{\lrcorner})^G_{DGA}\text{-Mod}$$

Step 4: Rigidity

$(R^{\lrcorner})^G_{DGA}$ is intrinsically formal.

1. $\pi_*(\widetilde{E}\mathcal{F})^G \cong \pi_* S^0 \cong \mathbb{Q}[0]$.

2. Note $(DEG_+)^G = \mathrm{Hom}(EG_+, S^0)^G \simeq \mathrm{Hom}(BG_+, S^0)$.
So $\pi_*(DE\mathcal{F}_+)^G \cong \prod_F H^*(BG/F) \cong \prod_F \mathbb{Q}[x_F]$

3. $\pi_*(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^G \cong \mathcal{E}_G^{-1} \prod_F \mathbb{Q}[x_F]$

Thus $(R^{\lrcorner})^G_{DGA}$ is quasi-isomorphic to $R^{\lrcorner}_{alg} = H_*(R^{\lrcorner})^G_{DGA}$:

$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \prod_F \mathbb{Q}[x_F] & \longrightarrow & \mathcal{E}_G^{-1} \prod_F \mathbb{Q}[x_F] \end{array}$$

Summary:

$$S^0\text{-Mod}_G \simeq_1 \text{cell-}R^{\lrcorner}\text{-Mod}_G \simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod}$$
$$\simeq_3 \text{cell-d.g.-}(R^{\lrcorner})^G_{DGA}\text{-Mod} \simeq_4 \text{cell-d.g.-}R^{\lrcorner}_{alg}\text{-Mod}$$

Step 5: Small algebraic model.

For $G = SO(2)$, $\mathcal{A}(G)$ is the category of d.g. modules $N \to M \leftarrow V$ over

$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \vartheta_{\mathcal{F}} & \longrightarrow & \mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \end{array}$$

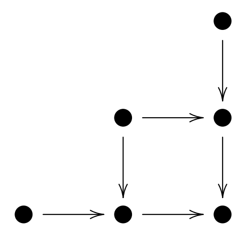
such that both structure maps are isomorphisms.

1. Quasi-coherence: $\mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \otimes_{\vartheta_{\mathcal{F}}} N \cong \mathcal{E}_G^{-1} N \xrightarrow{\cong} M$.
2. Extended: $\mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \otimes_{\mathbb{Q}} V \cong M$.

$$\simeq_4 \text{ cell-d.g.-}R_{alg}^{\lrcorner}\text{-Mod} \simeq_5 \text{d.g.}\mathcal{A}(G)$$

Theorem 1. For $G = SO(2)$, the homotopy theory of rational G-spectra is modeled by the abelian category $\mathcal{A}(G)$. Here $\mathcal{A}(G)$ has injective dimension one.

General outline is the same for all tori, just have larger diagrams. For G a 2-torus, the diagram shape is:



For an n-torus there are n layers.

Can restrict to families of fixed points.

For example, free G-spectra with $G = SO(2)$: have a module N over $H^*(BG)$, with $V = 0, M = 0$. The quasi-coherence condition says $\mathcal{E}_G^{-1}N \cong M = 0$; that is, N is torsion.

Theorem 2. The homotopy theory of free rational G-spectra is modeled by torsion modules over H^*BG.