A monoidal algebraic model for free rational torus-equivariant spectra

Joint work with J. Greenlees

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Thm.(Gabriel)
Let $\mathcal{C}$ be a cocomplete, abelian category with a small projective generator G. Let $\mathcal{E}(G) = \mathcal{C}(G, G)$ be the endomorphism ring of G. Then

$$\mathcal{C} \cong \text{Mod-}\mathcal{E}(G)$$

Differential graded Morita equivalence

Defn: $\mathcal{C}$ is a Ch_R-*model category* if it is enriched and tensored over Ch_R in a way that is compatible with the model structures.

Example: differential graded modules over a dga. Note, $\mathcal{E}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ is a dga.

Defn: An object X is *small* in $\mathcal{C}$ if $\oplus[X, A_i] \to [X, \coprod A_i]$ is an isomorphism.

An object X is a *generator* of $\mathcal{C}$ (or $\mathcal{H}o(\mathcal{C})$) if the only localizing subcategory containing X is $\mathcal{H}o(\mathcal{C})$ itself. (A *localizing* subcategory is a triangulated subcategory which is closed under coproducts.)

Example: A is a small generator of A-Mod.

Thm: If $\mathcal{C}$ is a Ch_R-model category with a (cofibrant and fibrant) small generator G then $\mathcal{C}$ is Quillen equivalent to (right) d.g. modules over $\mathcal{E}(G)$.

$$\mathcal{C} \simeq_Q \mathrm{Mod}\text{-}\mathcal{E}(G)$$

Example: Koszul duality

Consider the graded ring $P_{\mathbb{Q}}[c]$ with $|c| = -2$.
Let tor P-Mod be d.g. torsion $P_{\mathbb{Q}}[c]$-modules.

$\mathbb{Q}[0]$ is a small generator of tor P-Mod.
Let $\widetilde{Q}$ be a cofibrant and fibrant replacement.

Corollary: There is a Quillen equivalence:

$$\text{tor}\, P\text{-Mod} \simeq_Q \text{Mod-}\mathcal{E}(\widetilde{Q})$$

$$- \otimes_{\mathcal{E}(\widetilde{Q})} \widetilde{Q} : \text{Mod-}\mathcal{E}(\widetilde{Q}) \rightleftarrows \text{tor}\, P\text{-Mod} : \text{Hom}_{P[c]}(\widetilde{Q}, -)$$

I gave a sketch of the proof of this result: The right and left adjoints preserve the generators. They also preserve coproducts and triangles (they are exact), so they induce equivalences on the homotopy category. The same proof works in all cases of Morita equivalences in the rest of the talk.

Note $\mathcal{E}(\widetilde{Q}) = \text{Hom}_{P[c]}(\widetilde{Q}, \widetilde{Q})$
$\simeq \Lambda_{\mathbb{Q}}[x]$ with $|x| = 1$.

Corollary: Extension and restriction of scalars induce another Quillen equivalence:

$$- \otimes_{\mathcal{E}(\widetilde{Q})} \Lambda_{\mathbb{Q}}[x] : \text{Mod-}\mathcal{E}(\widetilde{Q}) \rightleftarrows \text{Mod-}\Lambda_{\mathbb{Q}}[x] : \text{res}.$$

Morita Equivalence over spectra

Defn: Let Sp denote a monoidal model category of spectra. $\mathcal{C}$ is a Sp-*model category* if it is compatibly enriched and tensored over Sp. $\mathcal{E}(X) = F_{\mathcal{C}}(X, X)$ is a ring spectrum.

Thm: (Schwede-S.) If $\mathcal{C}$ is a Sp-model category with a (cofibrant and fibrant) small generator G then $\mathcal{C}$ is Quillen equivalent to (right) module spectra over $\mathcal{E}(G) = F_{\mathcal{C}}(G, G)$.

$$\mathcal{C} \simeq_Q \text{Mod-}\,\mathcal{E}(G)$$

$$- \otimes_{\mathcal{E}(G)} G : \text{Mod-}\,\mathcal{E}(G) \rightleftarrows \mathcal{C} : F_{\mathcal{C}}(G, -)$$

Thm:(Dugger) Any combinatorial, stable model category is Quillen equivalent to a Sp^{Σ}-model category. Lurie also has results along these lines, but for quasi-categories (or infinity-categories) instead of model categories.

Rational stable model categories

Defn: A Sp-model category is rational if $[X, Y]_{\mathcal{C}}$ is a rational vector space for all X, Y in $\mathcal{C}$. In this case $\mathcal{E}(X) = F_{\mathcal{C}}(X, X) \simeq H\mathbb{Q} \wedge cF_{\mathcal{C}}(X, X)$.

Rational spectral algebra $\simeq$ d.g. algebra:

- There are composite Quillen equivalences
$$\Theta : H\mathbb{Q}\text{-Alg} \rightleftarrows \text{DGA}_{\mathbb{Q}} : \mathbb{H}.$$
- For any $H\mathbb{Q}$-algebra spectrum B,
$$\text{Mod-}B \rightleftarrows \text{Mod-}\Theta B.$$

Thm: If $\mathcal{C}$ is a rational Sp-model category with a (cofibrant and fibrant) small generator G then there are Quillen equivalences:

$$\begin{aligned}\mathcal{C} &\simeq_Q \text{Mod-}\mathcal{E}(G)\\ &\simeq_Q \text{Mod-}(H\mathbb{Q} \wedge c\mathcal{E}(G))\\ &\simeq_Q \text{Mod-}\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G)).\end{aligned}$$

$\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G))$ is a dga.

Free rational $\mathbb{S}^1$-equivariant spectra

Let $\mathcal{F}_{\mathbb{S}^1}$ denote free rational $\mathbb{S}^1$-equivariant spectra (a Sp-model category).

Note $S_{\mathbb{Q}}[\mathbb{S}^1] = H\mathbb{Q} \wedge \Sigma^\infty \mathbb{S}^1_+$ is a generator of $\mathcal{F}_{\mathbb{S}^1}$.

Also, $\mathcal{E}(S_{\mathbb{Q}}[\mathbb{S}^1]) = F_{\mathcal{F}_{\mathbb{S}^1}}(S_{\mathbb{Q}}[\mathbb{S}^1], S_{\mathbb{Q}}[\mathbb{S}^1]) \simeq S_{\mathbb{Q}}[\mathbb{S}^1]$

Corollary: There are Quillen equivalences:

$$\mathcal{F}_{\mathbb{S}^1} \simeq_Q \text{Mod-}\, S_{\mathbb{Q}}[\mathbb{S}^1] \simeq_Q \text{Mod-}\, \mathcal{C}_*(\mathbb{S}^1)$$

where $\mathcal{C}_*(\mathbb{S}^1)$ is the dga $\Theta(cS_{\mathbb{Q}}[\mathbb{S}^1])$.

Recall Koszul duality:

$$\text{Mod-}\, \Lambda_{\mathbb{Q}}[x] \rightleftarrows \text{tor}\, P[c]\,\text{-Mod}$$

$$\text{Mod-}\, \mathcal{C}_*(\mathbb{S}^1) \rightleftarrows \text{tor}\, \mathcal{C}^*(B\mathbb{S}^1)\,\text{-Mod}$$

$P[c]$ is intrinsically formal; $P[c] \to \mathcal{C}^*(B\mathbb{S}^1)$

Thm:

$$\mathcal{F}_{\mathbb{S}^1} \simeq_Q \text{tor}\, P[c]\,\text{-Mod} \simeq_Q \text{Mod-}\, \Lambda_{\mathbb{Q}}[x]$$

Free rational $\mathbb{T}$-equivariant spectra

Let $\mathcal{F}_{\mathbb{T}}$ denote free rational $\mathbb{T}$-equivariant spectra where $\mathbb{T}$ is the rank r torus. (a Sp-model category)

Note $S_{\mathbb{Q}}[\mathbb{T}] = H\mathbb{Q} \wedge \Sigma^{\infty}\mathbb{T}_+$ is a generator of $\mathcal{F}_{\mathbb{T}}$.
Also, $\mathcal{E}(S_{\mathbb{Q}}[\mathbb{T}]) = F_{\mathcal{F}_{\mathbb{T}}}(S_{\mathbb{Q}}[\mathbb{T}], S_{\mathbb{Q}}[\mathbb{T}]) \simeq S_{\mathbb{Q}}[\mathbb{T}]$

Corollary: There are Quillen equivalences:

$$\mathcal{F}_{\mathbb{T}} \simeq_Q \text{Mod-}\, S_{\mathbb{Q}}[\mathbb{T}] \simeq_Q \text{Mod-}\, \mathcal{C}_*(\mathbb{T})$$

where $\mathcal{C}_*(\mathbb{T})$ is the dga $\Theta(cS_{\mathbb{Q}}[\mathbb{T}])$.

Again have Koszul duality:

$$\text{Mod-}\, \Lambda_{\mathbb{Q}}[x_1, \cdots x_r] \rightleftarrows \text{tor}\, P[c_1, \cdots c_r]\text{-Mod}$$

$$\text{Mod-}\, \mathcal{C}_*(\mathbb{T}) \rightleftarrows \text{tor}\, \mathcal{C}^*(B\mathbb{T})\text{-Mod}$$

$P[c_1, \cdots , c_r]$ is intrinsically formal as a commutative $\mathbb{Q}$ − DGA.

Thm: If $\mathcal{C}^*(B\mathbb{T})$ is commutative, then
$\mathcal{F}_{\mathbb{T}} \simeq_Q \text{tor}\, P[c_1, \cdots c_r]\text{-Mod} \simeq_Q \text{Mod-}\, \Lambda_{\mathbb{Q}}[x_1, \cdots x_r]$

Many generators

Defn: If $\mathcal{C}$ is a Ch_R-model category with a set of generators $\mathcal{G}$, then $\mathcal{E}(\mathcal{G})$ is the enriched subcategory of $\mathcal{C}$ with object set $\mathcal{G}$.

A (right) module over $\mathcal{E}(\mathcal{G})$ is a Ch_R-enriched functor from $\mathcal{E}(\mathcal{G})^{op}$ to Ch_R.

Here I drew a picture of the example below.

Example:
If $\mathcal{G} = \{G, H\}$, then a module over $\mathcal{E}(\mathcal{G})$ consists of

- $M(G)$ an $\mathcal{E}(G) = \mathrm{Hom}_{\mathcal{C}}(G, G)$-module,
- $M(H)$ an $\mathcal{E}(H) = \mathrm{Hom}_{\mathcal{C}}(H, H)$-module,
- $\alpha : M(H) \otimes \mathrm{Hom}_{\mathcal{C}}(G, H) \to M(G)$ and
- $\beta : M(G) \otimes \mathrm{Hom}_{\mathcal{C}}(H, G) \to M(H)$

with certain compatibility properties.

Many generators Morita equivalence

Example: For each $K \in \mathcal{G}$,
there is a representable module $F_K = \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, K)$.

The set $\{F_K\}_{K \in \mathcal{G}}$ generates Mod-$\mathcal{E}(\mathcal{G})$.

Thm: If $\mathcal{C}$ is a Ch_R-model category with a set of (cofibrant and fibrant) small generators $\mathcal{G}$ then $\mathcal{C}$ is Quillen equivalent to (right) modules over $\mathcal{E}(\mathcal{G})$.

$$\mathcal{C} \simeq_Q \text{Mod-}\mathcal{E}(\mathcal{G})$$

Monoidal structure on Mod-$\mathcal{E}(\mathcal{G})$

Consider:
$(\mathcal{C}, \otimes)$ a symmetric monoidal Ch_R-model category
$\mathcal{G}$ a set in $\mathcal{C}$ which is closed under $\otimes$

Then $\mathcal{E}(\mathcal{G})$ is a symmetric monoidal Ch_R-category.

Examples

- $\mathcal{G} = \{G^{\otimes n}\}, G^{\otimes 0} = \mathbb{I}_{\mathcal{C}}$
- $\mathcal{G} = \{\mathbb{I}_{\mathcal{C}}\}$

Prop:(Day) For $\mathcal{E}(\mathcal{G})$ as above, Mod-$\mathcal{E}(\mathcal{G})$ is also a symmetric monoidal category.

Defn: Given M, N in Mod-$\mathcal{E}(\mathcal{G})$,
define $M\overline{\otimes}N$ on $\mathcal{G} \times \mathcal{G}$ by:

$$M\overline{\otimes}N(G, H) = M(G) \otimes_R N(H).$$

Define $M\Box_{\mathcal{E}}N$ in Mod-$\mathcal{E}(\mathcal{G})$ as the left Kan extension of $M\overline{\otimes}N$ over $\mathcal{G} \times \mathcal{G} \xrightarrow{\otimes} \mathcal{G}$.

Example:

$$F_G \Box_{\mathcal{E}} F_H = F_{G \otimes H}$$

Monoidal Morita Equivalence

Thm: Let $\mathcal{C}$ be a symmetric monoidal Ch_R-model category and $\mathcal{G}$ be a set of cofibrant and fibrant, small generators which is closed under the product. Then there is a monoidal Quillen equivalence

$$(\mathcal{C}, \otimes) \simeq_Q (\text{Mod-}\,\mathcal{E}(\mathcal{G}), \Box_{\mathcal{E}}).$$

The left adjoint is strong symmetric monoidal and the right adjoint is lax symmetric monoidal.

Prf: As above, the left adjoint takes generators to generators, $L(F_G) \cong G$.
So, $L(F_G) \otimes L(F_H) \cong G \otimes H$
and $L(F_G \Box_{\mathcal{E}} F_H) \cong L(F_{G \otimes H}) \cong G \otimes H$.

Thm: There are monoidal Quillen equivalences
$\mathcal{F}_{\mathbb{T}} \simeq_Q \text{tor}\, P[c_1, \cdots c_r]\text{-Mod} \simeq_Q \text{Mod-}\,\Lambda_{\mathbb{Q}}[x_1, \cdots x_r]$

General Conclusion

Thm. Rational $\mathbb{T}$-equivariant spectra has a *small* monoidal algebraic model.

Prf. Preprint availabe on my web page. There will be future drafts.

Note that *free* does not appear in the above theorem.