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THE CELLULARIZATION PRINCIPLE FOR QUILLEN ADJUNCTIONS

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(*communicated by Daniel Dugger*)

Abstract

The Cellularization Principle states that under rather weak conditions, a Quillen adjunction of stable model categories induces a Quillen equivalence on cellularizations provided there is a derived equivalence on cells. We give a proof together with a range of examples.

1. Introduction

The purpose of this paper is to publicize a useful general principle when comparing model categories: whenever one has a Quillen adjunction

$$F\colon \mathbb{M} \rightleftarrows \mathbb{N} : U$$

comparing two stable model categories, we obtain another Quillen adjunction by cellularizing the two model categories with respect to corresponding objects. Furthermore we obtain a Quillen *equivalence* provided the cells are small and the derived unit or counit is an equivalence on cells. In this case, the cellularization of the adjunction induces a homotopy category level equivalence between the respective localizing subcategories. The hypotheses are mild, and the statement may appear like a tautology. The Cellularization Principle can be directly compared to another extremely powerful formality, that a natural transformation of cohomology theories that is an isomorphism on spheres is an equivalence.

This result was first proved in an appendix of the original versions of [**4**], but the range of cases where the conclusion is useful led us to present the result separately from that particular application.

The paper is layed out as follows: in Section 2 we give the statement and proof of the Cellularization Principle, and the following sections give a selection of examples.

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2. Cellularization of model categories

Throughout the paper we need to consider models for categories of cellular objects, thought of as built from a set of basic cells using coproducts and cofibre sequences. These models are usually obtained by the process of cellularization (sometimes known as colocalization or right localization) of model categories, with the cellular objects appearing as the cofibrant objects. Because it is fundamental to our work, we recall some of the basic definitions from [**8**].

Definition 2.1. [**8**, 3.1.8] Let $\mathbb{M}$ be a model category and $\mathcal{K}$ be a set of objects in $\mathbb{M}$. A map $f\colon X \to Y$ is a *$\mathcal{K}$-cellular equivalence* if for every element A in $\mathcal{K}$ the induced map of homotopy function complexes [**8**, 17.4.2] $f_*\colon \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$ is a weak equivalence. A cofibrant object W is *$\mathcal{K}$-cellular* if $f_*\colon \mathrm{map}(W, X) \to \mathrm{map}(W, Y)$ is a weak equivalence for any $\mathcal{K}$-cellular equivalence f.

One can cellularize a right proper model category under very mild finiteness hypotheses. To avoid confusion due to the dual use of the word "cellular" we recall that a *cellular* model category is a cofibrantly generated model category with smallness conditions on its generating cofibrations and acyclic cofibrations [**8**, 12.1.1].

Proposition 2.2. *[**8**, 5.1.1] Let $\mathbb{M}$ be a right proper, cellular model category and let $\mathcal{K}$ be a set of objects in $\mathbb{M}$. The $\mathcal{K}$-cellularized model category $\mathcal{K}$-cell-$\mathbb{M}$ exists: it has the same underlying category as $\mathbb{M}$, its weak equivalences are the $\mathcal{K}$-cellular equivalences, the fibrations the same as in the original model structure on $\mathbb{M}$, and the cofibrations are the maps with the left lifting property with respect to the trivial fibrations. The cofibrant objects are the $\mathcal{K}$-cellular objects.*

Remark 2.3. Since the $\mathcal{K}$-cellular equivalences are defined using homotopy function complexes, the $\mathcal{K}$-cellularized model category $\mathcal{K}$-cell-$\mathbb{M}$ depends only on the homotopy type of the objects in $\mathcal{K}$.

It is useful to have the following further characterization of the cofibrant objects.

Proposition 2.4. *[**8**, 5.1.5] If $\mathcal{K}$ is a set of cofibrant objects in $\mathbb{M}$, then the class of $\mathcal{K}$-cellular objects agrees with the smallest class of cofibrant ojects in $\mathbb{M}$ that contains $\mathcal{K}$ and is closed under homotopy colimits and weak equivalences.*

Throughout this paper we consider stable cellularizations of stable model categories. Say that a set $\mathcal{K}$ is *stable* if for any $A \in \mathcal{K}$ all of its suspensions (and desuspensions) are also in $\mathcal{K}$ up to weak equivalence. That is, since the cellularization only depends on the homotopy type of elements in $\mathcal{K}$, if $A \in \mathcal{K}$, then for all $i \in \mathbb{Z}$ there are objects $B_i \in \mathcal{K}$ with $B_i \simeq \Sigma^i A$. In this case, for $\mathbb{M}$ a stable model category and $\mathcal{K}$ a stable set of objects, $\mathcal{K}$-cell-$\mathbb{M}$ is again a stable model category; see [**1**, 4.6]. In this case, one can use homotopy classes of maps instead of the homotopy function complexes in Definition 2.1. That is, a map $f\colon X \to Y$ is a $\mathcal{K}$-cellular equivalence if and only if for every element A in $\mathcal{K}$ the induced map $[A, X]_* \to [A, Y]_*$ is an isomorphism; see [**1**, 4.4].

Proposition 2.5. *If $\mathbb{M}$ is a right proper, stable, cellular model category and $\mathcal{K}$ is stable, then $\mathcal{K}$ detects trivial objects. That is, an object X is trivial in* $\mathrm{Ho}(\mathcal{K}$*-cell-*$\mathbb{M})$*, if and only if for each element A in $\mathcal{K}$, $[A, X]_* = 0$, and this group of graded morphisms can equally be calculated in the homotopy category of $\mathbb{M}$.*

Proof. By [**9**, 7.3.1], the set of cofibres of the generating cofibrations detects trivial objects. In this stable situation, a set of generating cofibrations is produced as follows in [**1**, 4.9]. Define a set of horns on $\mathcal{K}$ by

$$\Lambda\mathcal{K} = \{X \otimes \partial\Delta[n]_+ \to X \otimes \Delta[n]_+ | n \geqslant 0, X \in \mathcal{K}\}$$

where here $\otimes$ is defined using framings (see [**1**], [**9**], or [**8**]). If J is the set of generating acyclic cofibrations in $\mathbb{M}$, then $J \cup \Lambda\mathcal{K}$ is the set of generating cofibrations for $\mathcal{K}$-cell-$\mathbb{M}$. Since $\mathcal{K}$ is stable, the cofibres of these maps are either contractible or are weakly equivalent to objects in $\mathcal{K}$ again. □

Under a finiteness condition $\mathcal{K}$ is also a set of generators. An object K is small in the homotopy category (from now on simply *small*[1]) if, for any set of objects $\{Y_\alpha\}$, the natural map $\bigoplus_\alpha [K, Y_\alpha] \longrightarrow [K, \bigvee_\alpha Y_\alpha]$ is an isomorphism.

Corollary 2.6. *[**12**, 2.2.1] If $\mathbb{M}$ is a right proper, stable, cellular model category and $\mathcal{K}$ is a stable set of small objects, then $\mathcal{K}$ is a set of generators of* Ho($\mathcal{K}$*-cell-*$\mathbb{M}$). *That is, the only localizing subcategory containing $\mathcal{K}$ is* Ho($\mathcal{K}$*-cell-*$\mathbb{M}$) *itself.*

Our main theorem states that given a Quillen pair, appropriate cellularizations of model categories preserve Quillen adjunctions and induce Quillen equivalences.

Theorem 2.7. (The Cellularization Principle.) *Let $\mathbb{M}$ and $\mathbb{N}$ be right proper, stable, cellular model categories with $F\colon \mathbb{M} \to \mathbb{N}$ a left Quillen functor with right adjoint U. Let Q be a cofibrant replacement functor in $\mathbb{M}$ and R a fibrant replacement functor in $\mathbb{N}$.*

1. *Let $\mathcal{K} = \{A_\alpha\}$ be a set of objects in $\mathbb{M}$ with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set in $\mathbb{N}$. Then F and U induce a Quillen adjunction*

$$F\colon \mathcal{K}\text{-cell-}\mathbb{M} \rightleftarrows FQ\mathcal{K}\text{-cell-}\mathbb{N} : U$$

   between the $\mathcal{K}$-cellularization of $\mathbb{M}$ and the $FQ\mathcal{K}$-cellularization of $\mathbb{N}$.

2. *If $\mathcal{K} = \{A_\alpha\}$ is a stable set of small objects in $\mathbb{M}$ such that for each A in $\mathcal{K}$ the object FQA is small in $\mathbb{N}$ and the derived unit $QA \to URFQA$ is a weak equivalence in $\mathbb{M}$, then F and U induce a Quillen equivalence between the cellularizations:*

$$\mathcal{K}\text{-cell-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-cell-}\mathbb{N}.$$

3. *If $\mathcal{L} = \{B_\beta\}$ is a stable set of small objects in $\mathbb{N}$ such that for each B in $\mathcal{L}$ the object URB is small in $\mathbb{M}$ and the derived counit $FQURB \to RB$ is a weak equivalence in $\mathbb{N}$, then F and U induce a Quillen equivalence between the cellularizations:*

$$UR\mathcal{L}\text{-cell-}\mathbb{M} \simeq_Q \mathcal{L}\text{-cell-}\mathbb{N}.$$

Proof. Using the equivalences in [**8**, 3.1.6], the criterion in [**8**, 3.3.18(2)] (see also [**10**, 2.2]) for showing that F and U induce a Quillen adjoint pair on the cellularized model categories in (1) is equivalent to requiring that U takes $FQ\mathcal{K}$-cellular equivalences between fibrant objects to $\mathcal{K}$-cellular equivalences. Any Quillen adjunction induces

[1] some authors use 'compact' for this notion

a weak equivalence $\mathrm{map}(A, URX) \simeq \mathrm{map}(FQA, X)$ of the homotopy function complexes, see for example [**8**, 17.4.15]. So $f\colon X \to Y$ induces a weak equivalence $f_*\colon \mathrm{map}(FQA, X) \to \mathrm{map}(FQA, Y)$ if and only if $Uf_*\colon \mathrm{map}(A, URX) \to \mathrm{map}(A, URY)$ is a weak equivalence. Thus in (1), U preserves (and reflects) the cellular equivalences between fibrant objects. Hence, U induces a Quillen adjunction on the cellularized model categories.

Similarly, $Uf_*\colon \mathrm{map}(URB, URX) \to \mathrm{map}(URB, URY)$ is a weak equivalence if and only if $f_*\colon \mathrm{map}(FQURB, X) \to \mathrm{map}(FQURB, Y)$ is. Given the hypothesis in (3) that $FQURB \to RB$ is a weak equivalence, it follows that Uf_* is a weak equivalence if and only if $f_*\colon \mathrm{map}(B, X) \to \mathrm{map}(B, Y)$ is. Thus, it follows in (3) that U preserves (and reflects) the cellular equivalences between fibrant objects. Hence, U induces a Quillen adjunction on the cellularized model categories. Note that the stability of $\mathbb{M}$, $\mathbb{N}$, $\mathcal{K}$ and $\mathcal{L}$ was not necessary for establishing the Quillen adjunction in (1) or (3).

We establish (2) in the next paragraph. One can make very similar arguments for (3) or one can deduce (3) from (2). To deduce (3) from (2), consider (2) applied to $\mathcal{K} = UR\mathcal{L}$. The hypothesis in (3) implies the hypothesis in (2) and thus produces a Quillen equivalence between $UR\mathcal{L}$-cell-$\mathbb{M}$ and $FQUR\mathcal{L}$-cell-$\mathbb{M}$. The hypothesis in (3) also implies that $FQUR\mathcal{L}$-cell-$\mathbb{M}$ and $\mathcal{L}$-cell-$\mathbb{M}$ are the same cellularization of $\mathbb{M}$. Thus, (3) follows.

We now return to the Quillen equivalence in (2). Since $\mathbb{M}$ and $\mathcal{K}$ are stable, $\mathcal{K}$-cell-$\mathbb{M}$ is a stable model category by [**1**, 4.6]. Since left Quillen functors preserve homotopy cofibre sequences, $FQ\mathcal{K}$, and hence also $FQ\mathcal{K}$-cell-$\mathbb{N}$, are stable. The Quillen adjunction in (1) induces a derived adjunction on the triangulated homotopy categories; we show that this is actually a derived equivalence. Both derived functors are exact (since the left adjoint commutes with suspension and cofibre sequences and the right adjoint commutes with loops and fibre sequences). As a left adjoint, F also preserves coproducts. We next show that the right adjoint preserves coproducts as well.

Since $\mathcal{K} = \{A_\alpha\}$ detects $\mathcal{K}$-cellular equivalences, to show that U preserves coproducts it suffices to show that for each $A_\alpha \in \mathcal{K}$ and any family $\{X_i\}$ of objects in $\mathbb{N}$ the natural map

$$[A_\alpha, \bigvee_i UX_i] \to [A_\alpha, U(\bigvee_i X_i)]$$

is an isomorphism. Using the adjunction and the fact that each A_α is small, the source can be rewritten as

$$[A_\alpha, \bigvee_i UX_i] \cong \bigoplus_i [A_\alpha, UX_i] \cong \bigoplus_i [FQA_\alpha, X_i].$$

Similarly, using the adjunction, the target is isomorphic to $[FQA_\alpha, \bigvee_i X_i]$. Since FQA_α is assumed to be small, the source and target are isomorphic and this shows that U commutes with coproducts.

Consider the full subcategory of objects M in $\mathrm{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$ (or respectively, N in $\mathrm{Ho}(FQ\mathcal{K}\text{-cell-}\mathbb{N})$) such that the unit $QM \to URFQM$ (or respectively, the counit $FQURN \to RN$) of the adjunction is an equivalence. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. Since for each A in $\mathcal{K}$ the unit is an equivalence and $\mathcal{K}$ is a set of generators by Corollary 2.6, the unit is an equivalence on all of $\mathrm{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$. It follows that the counit is also an

equivalence for each object $N = FQA$ in $FQ\mathcal{K}$. Since $FQ\mathcal{K}$ is a set of generators for Ho($FQ\mathcal{K}$-cell-$\mathbb{N}$), the counit is also always an equivalence. Statement (2) follows. □

Note that if F and U form a Quillen equivalence on the original categories, then the conditions in Theorem 2.7 parts (2) and (3) are automatically satisfied. Thus, they also induce Quillen equivalences on the cellularizations.

Corollary 2.8. *Let $\mathbb{M}$ and $\mathbb{N}$ be right proper, stable cellular model categories with $F\colon \mathbb{M} \to \mathbb{N}$ a Quillen equivalence with right adjoint U. Let Q be a cofibrant replacement functor in $\mathbb{M}$ and R a fibrant replacement functor in $\mathbb{N}$.*

1. *Let $\mathcal{K} = \{A_\alpha\}$ be a stable set of small objects in $\mathbb{M}$, with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set of objects in $\mathbb{N}$. Then F and U induce a Quillen equivalence between the $\mathcal{K}$-cellularization of $\mathbb{M}$ and the $FQ\mathcal{K}$-cellularization of $\mathbb{N}$:*

$$\mathcal{K}\text{-cell-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-cell-}\mathbb{N}.$$

2. *Let $\mathcal{L} = \{B_\beta\}$ be a set of small objects in $\mathbb{N}$, with $UR\mathcal{L} = \{URB_\beta\}$ the corresponding set of objects in $\mathbb{N}$. Then F and U induce a Quillen equivalence between the $\mathcal{L}$-cellularization of $\mathbb{N}$ and the $UR\mathcal{L}$-cellularization of $\mathbb{M}$:*

$$UR\mathcal{L}\text{-cell-}\mathbb{M} \simeq_Q \mathcal{L}\text{-cell-}\mathbb{N}.$$

In [**10**, 2.3] Hovey gives criteria for when localizations preserve Quillen equivalences. Since cellularization is dual to localization, a generalization of this corollary without stability or smallness hypotheses follows from the dual of Hovey's statement.

3. Smashing localizations

We suppose given a map $\theta\colon R \longrightarrow T$ of ring spectra (or DGAs). This gives the extension and restriction of scalars Quillen adjunction

$$\theta_*\colon R\text{-mod} \rightleftarrows T\text{-mod} : \theta^* ,$$

where $\theta_* N = T \wedge_R N$. We apply Theorem 2.7 with $\mathbb{M} = R$-mod and $\mathbb{N} = T$-mod. The category of T-modules is generated by the T-module T, and we use that as the generating cell. The following uses the ideas of the Cellularization Principle.

Corollary 3.1. *If $T \wedge_R T \xrightarrow{\simeq} T$ is an equivalence of R-modules, then the Quillen pair*

$$\theta_*\colon T\text{-cell-}R\text{-modules} \rightleftarrows T\text{-modules} : \theta^*$$

induces

1. *a Quillen equivalence if $\theta^* T$ is small as an R-module, or*
2. *in general, an equivalence of triangulated categories*

$$T\text{-loc-Ho}(R\text{-modules}) \simeq \text{Ho}(T\text{-modules})$$

where loc denotes the localizing subcategory.

Proof. In the first case with $\theta^* T$ small, this follows directly from Part 3,Theorem 2.7 with $\mathbb{M} = R$-modules and $\mathbb{N} = T$-modules, taking T to be the generator of $\mathbb{N}$. In the second case, we again apply Part 3. Here the hypothesis shows that the counit is a

derived equivalence on cells. However the complication is that θ^*T will not usually be small as an R-module.

Nonetheless, the counit is still a derived equivalence for the R-module T. It remains to argue that the derived counit gives the stated equivalence. For this we note that the right adjoint preserves arbitrary sums: this is obvious if we are working with actual modules, but in general we may use the fact that θ^* is also a left adjoint (with right adjoint the coextension of scalars). It follows that we have an equivalence of the localizing subcategories generated by T on the two sides. □

For the first example, we take R and T to be conventional commutative rings or DGAs and $T = \mathcal{E}^{-1}R$ for some multiplicatively closed set $\mathcal{E}$. The condition is satisfied since $\mathcal{E}^{-1}R \otimes_R \mathcal{E}^{-1}R \cong \mathcal{E}^{-1}R$ and we find

$$\mathcal{E}^{-1}R\text{-loc-Ho}(R\text{-modules}) \simeq \text{Ho}(\mathcal{E}^{-1}R\text{-modules}).$$

Example 3.2. It is worth giving an example to show that we do not obtain a Quillen equivalence between T-cell-R-modules and T-modules in general. This shows that the smallness hypothesis in the Cellularization Principle is necessary.

For this we take $R = \mathbb{Z}$ and $T = \mathbb{Z}[1/p]$. We note that any object M in the localizing subcategory of T-cell-R-modules generated by T has the property that $M \simeq M[1/p]$. Accordingly $M = \mathbb{Z}/p^\infty$ is not in this localizing subcategory. On the other hand M is not T-cellularly equivalent to 0 since $\text{Hom}(\mathbb{Z}[1/p], \mathbb{Z}/p^\infty) \neq 0$.

More generally any smashing localization of module spectra behaves in a similar way (see [**7**] for related discussion). We first suppose that R is a cofibrant $\mathbb{S}$-algebra spectrum in the sense of [**3**], where $\mathbb{S}$ is the sphere spectrum; see also Remark 3.4 below. We next suppose given a smashing localization L of the category of R-module spectra. By [**3**, VIII.2.2], there is a ring map $\theta\colon R \longrightarrow LR$ so that we may apply the above discussion with $T = LR$. The smashing condition states that we have an equivalence $LN \simeq LR \wedge_R N$ for any R-module N.

Proposition 3.3. *Suppose given a cofibrant $\mathbb{S}$-algebra spectrum R and a smashing localization L on the category of R-modules. We have a diagram*

$$\begin{array}{ccc} R\text{-modules} & \longrightarrow & LR\text{-modules} \\ \downarrow & \nearrow_{\simeq} & \\ L(R\text{-modules}) & & \end{array}$$

of left Quillen functors where $L(R$-modules$)$ is the localization of the model category of R-modules. These induce a Quillen equivalence

$$L(R\text{-modules}) \simeq_Q LR\text{-modules},$$

and triangulated equivalences

$$LR\text{-loc-Ho }(R\text{-modules}) \simeq Ho(L(R\text{-modules})) \simeq Ho(LR\text{-modules}).$$

Proof. The ring map $\theta\colon R \longrightarrow LR = T$ gives the top horizontal left Quillen functor θ_*. The vertical map is the identity on underlying categories, which is a left Quillen functor by definition of the local model structure. The diagonal map is again θ_*, and

it exists because L is smashing so that L-equivalences are taken to equivalences of LR-modules. This gives the diagram of left Quillen functors.

To see the diagonal is a Quillen equivalence we apply Part 3 of the Cellularization Principle. The LR-module LR is a small generator of LR-modules, and the smashing condition means that $L(R\text{-modules})$ is generated by the single object LR, and the universal property together with the smallness of R shows LR is small in $L(R\text{-modules})$.

We apply Corollary 3.1 to obtain the statement about localizing categories, since the smashing condition applies to the generator $N = LR$ to show the hypotheses hold.

See also [**3**, VIII.3.2] and [**12**, 3.2(iii)]. □

Remark 3.4. For an arbitrary $\mathbb{S}$-algebra spectrum T, consider a cofibrant replacement $\theta\colon R \to T$ in $\mathbb{S}$-algebras and consider the relationship between smashing localizations of R-modules and T-modules. More precisely, suppose given a bifibrant T-module E so that L_E (localization with respect to the E-equivalences as in [**3**, VIII.1.1]) is a smashing localization on T-modules. In this case there is a corresponding R-module F, giving a smashing localization L_F on R-modules such that the localized model categories are Quillen equivalent,

$$L_F(R\text{-modules}) \simeq_Q L_E(T\text{-modules}).$$

In fact one can take F to be the cofibrant replacement in R-modules of $\theta^* E$. Since $\theta\colon R \to T$ is a weak equivalence, the functors θ_* and θ^* induce a Quillen equivalence between the categories of R-modules and T-modules. Using the criteria in [**10**, 2.3] for when localizations preserve Quillen equivalences (dual to Corollary 2.8 above), one can show that the localization model category of R-modules with respect to the F-equivalences is Quillen equivalent to the localization model category of T-modules with respect to the $\theta_* F \cong T \wedge_R F$-equivalences. Since $\theta_* F$ and E are weakly equivalent cofibrant T-modules, the $\theta_* F$-equivalences agree with the E-equivalences, so we have

$$L_F(R\text{-modules}) \simeq_Q L_{\theta_* F}(T\text{-modules}) \cong L_E(T\text{-modules}).$$

In this situation one can show that there is a weak equivalence $T \wedge_R L_F R \simeq L_E T$. Using this, one can then show that if L_E is smashing, then L_F is also a smashing localization. Note here though that [**3**, VIII.2.2] only applies to R. So although $L_F R$ is constructed as an S-algebra, $L_E T$ is only constructed as a T-module.

Perhaps the best known example in the category of spectra is when we consider the localization of $\mathbb{S}$-modules with respect to E_n, the nth p-local Morava E-theory. We denote localization with respect to any spectrum weakly equivalent to E_n by L_n. Following the remark above, we consider $R = c\mathbb{S}$, the cofibrant replacement of the sphere spectrum $\mathbb{S}$ and see that $L_n(\mathbb{S}\text{-modules})$ and $L_n(c\mathbb{S}\text{-modules})$ are Quillen equivalent. Proposition 3.3 then shows that

$$L_n(\mathbb{S}\text{-modules}) \simeq_Q L_n(c\mathbb{S}\text{-modules}) \simeq_Q (L_n c\mathbb{S})\text{-modules}$$

and the homotopy categories of these are equivalent to $(L_n\mathbb{S})$-loc-Ho($\mathbb{S}$-modules).

4. Isotropic equivalences of ring G-spectra

We suppose given a map $\theta\colon R \longrightarrow T$ of ring G-spectra. This gives the extension and restriction of scalars Quillen adjunction

$$\theta_*\colon R\text{-mod} \rightleftarrows T\text{-mod} : \theta^* .$$

We apply Theorem 2.7 Part 2 with $\mathbb{M} = R$-mod and $\mathbb{N} = T$-mod. If G is the trivial group, then R generates R-modules and the Cellularization Principle shows we have an equivalence if θ is a weak equivalence of R-modules.

If G is non-trivial, we get a somewhat more interesting example. The category of R-modules is generated by the extended objects $G/H_+ \wedge R$ as H runs through closed subgroups of G and the unit is the comparison $G/H_+ \wedge \theta$. If $\mathcal{F}$ is a family of subgroups, we say θ is an $\mathcal{F}$-equivalence if $G/H_+ \wedge \theta$ is an equivalence for all H in $\mathcal{F}$. Define $\mathcal{F}$-cellularization of R-mod to be cellularization with respect to the set of all suspensions and desuspensions of objects $G/H_+ \wedge R$ for H in $\mathcal{F}$. Then the Cellularization Principle shows that if θ is an $\mathcal{F}$-equivalence we have a Quillen equivalence

$$\mathcal{F}\text{-cell-}R\text{-module-}G\text{-spectra} \simeq \mathcal{F}\text{-cell-}T\text{-module-}G\text{-spectra}.$$

5. Torsion modules

Let R be a conventional commutative Noetherian ring and I an ideal. We apply Theorem 2.7, with $\mathbb{N}$ the category of differential graded R-modules and $\mathbb{M}$ the category of differential graded I-power torsion modules. There is an adjunction

$$i\colon I\text{-power-torsion-}R\text{-modules} \rightleftarrows R\text{-modules} : \Gamma_I$$

with left adjoint i the inclusion and the right adjoint Γ_I defined by

$$\Gamma_I(M) = \{m \in M \mid I^N m = 0 \text{ for } N >> 0\}.$$

Both of these categories support injective model structures by [**9**, 2.3.13], with cofibrations the monomorphisms and weak equivalences the quasi-isomorphisms. For torsion-modules, one needs to bear in mind that to construct products and inverse limits one forms them in the category of all R-modules and then applies the right adjoint Γ_I; see also [**5**, 8.6]. With these structures the above adjunction is a Quillen adjunction.

We now consider the Cellularization Principle with $\mathbb{M}$ the I-power-torsion-R-modules and $\mathbb{N}$ the R-modules. If $I = (x_1, x_2, \dots, x_n)$, we may form the Koszul complex $K := K(x_1, x_2, \dots, x_n)$ as the tensor product of the complexes $R \xrightarrow{x_i} R$, noting that it is small by construction and therefore suitable for use as a cell. Since the homology of K is I-power torsion, K is equivalent to an object K' in the category of I-power torsion modules, which we may take to be fibrant. We now apply the Cellularization Principle to give an equivalence

$$\Gamma_I K'\text{-cell-}I\text{-power-torsion-}R\text{-modules} \simeq K'\text{-cell-}R\text{-modules}.$$

It is proved in [**2**, 6.1] that the localizing subcategory generated by R/I is also generated by $K \simeq K'$. By the same proof, we see that $K' = \Gamma_I K'$ generates the category of I-power torsion modules and we conclude

$$I\text{-power-torsion-}R\text{-modules} \simeq R/I\text{-cell-}R\text{-modules}.$$

6. Hasse equivalences

The idea here is that if a ring (spectrum or differential graded algebra) R is expressed as the pullback of a diagram of rings, the Cellularization Principle lets us build up the model category of differential graded R-modules from categories of modules over the terms. See also [**6**] for a more general treatment. We apply the standard context of Theorem 2.7 with $\mathbb{M}$ the category of R-modules.

6.1. Diagrams of modules

To describe $\mathbb{N}$ we start with a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R^l \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ R^c & \xrightarrow[\delta]{} & R^t \end{array}$$

of rings.

Example 6.1. The classical Hasse principle is built on the pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}^\wedge_p & \longrightarrow & (\prod_p \mathbb{Z}^\wedge_p) \otimes \mathbb{Q}. \end{array}$$

Returning to the general case, we delete R and consider the diagram

$$R^\lrcorner = \left(\begin{array}{ccc} & & R^l \\ & & \downarrow{\scriptstyle g} \\ R^c & \xrightarrow[d]{} & R^t \end{array} \right)$$

with three objects. We may form the category $\mathbb{N} = R^\lrcorner$-mod of diagrams

$$\begin{array}{ccc} & & M^l \\ & & \downarrow{\scriptstyle h} \\ M^c & \xrightarrow[e]{} & M^t \end{array}$$

where M^l is an R^l-module, M^c is an R^c-module, M^t is an R^t-module and the maps h and e are module maps over the corresponding maps of rings. That is, $h\colon M^l \to g^*M^t$ is a map of R^l-modules and $e\colon M^c \to d^*M^t$ is a map of R^c-modules. We will return to model structures below.

6.2. An adjoint pair

Since $R^\lrcorner$ is a diagram of R-algebras, termwise tensor product gives a functor

$$R^\lrcorner \otimes_R (\cdot)\colon R\text{-mod} \longrightarrow R^\lrcorner\text{-mod}.$$

Similarly, since R maps to the pullback $PR^\lrcorner$, pullback gives a functor

$$P\colon R^\lrcorner\text{-mod} \longrightarrow R\text{-mod}.$$

It is easily verified that these give an adjoint pair

$$R^{\lrcorner} \otimes_R (\cdot)\colon R\text{-mod} \rightleftarrows R^{\lrcorner}\text{-mod} : P\ .$$

We may then consider the derived unit

$$\eta\colon M \longrightarrow \underline{P}(R^{\lrcorner} \otimes^L_R M).$$

Since R is the generator of the category of R-modules, we want to require that η is an equivalence when $M = R$, which is to say the original diagram of rings is a homotopy pullback. In fact, we first fibrantly replace the original diagram of rings in the diagram-injective model category of pull-back diagrams of rings. See [**9**, 5.1.3] or [**8**, 15.3.4]. This is discussed in more detail in [**6**].

On the other hand, we cannot expect the counit of the adjunction to be an equivalence since we can add any module to M^t without changing $PM^{\lrcorner}$. This is where the Cellularization Principle comes in. We should use the image of R to cellularize the category of diagrams of modules. In preparation for this, we describe the model structure.

6.3. Model structures

We give categories of (differential graded) modules over a ring the (algebraically) projective model structure, with homology isomorphisms as weak equivalences and fibrations the surjections. The cofibrations are retracts of relative cell complexes, where the spheres are shifted copies of R. The category $R^{\lrcorner}$-mod gets the diagram-injective model structure in which cofibrations and weak equivalences are maps which have this property objectwise; the fibrant objects have γ and δ surjective. This diagram-injective model structure is shown to exist for more general diagrams of ring spectra in an appendix of the original versions of [**4**], see also [**6**], and the same proof works for DGAs.

6.4. The Quillen equivalence

Since extension of scalars is a left Quillen functor for the (algebraically) projective model structure for any map of DGAs, $R^{\lrcorner} \otimes_R -$ preserves cofibrations and weak equivalences and is therefore also a left Quillen functor. We then apply the Cellularization Principle to obtain the following result.

Proposition 6.2. *Assume given a commutative square of DGAs which is a homotopy pullback. The adjunction induces a Quillen equivalence*

$$R\text{-}mod \xrightarrow{\simeq} R^{\lrcorner}\text{-}cell\text{-}R^{\lrcorner}\text{-}mod,$$

where cellularization is with respect to the image, $R^{\lrcorner}$, of the generating R-module R.

Proof. We apply Theorem 2.7, which states that if we cellularize the model categories with respect to corresponding sets of small objects, we obtain a Quillen adjunction.

In the present case, we cellularize with respect to the single small R-module R on the left, and the corresponding diagram $R^{\lrcorner}$ on the right. First we verify that $R^{\lrcorner}$ is small. Consider the three evaluation functors from $R^{\lrcorner}$-modules down to modules over the rings R^l, R^c, or R^t and the associated left adjoints of these evaluation functors

L^l, L^c, and L^t. The $R^\lrcorner$-module $R^\lrcorner$ is the pushout of the following diagram.

$$\begin{array}{ccc} & & L^lR^l \\ & & \uparrow \\ L^cR^c & \longleftarrow & L^tR^t \end{array}$$

Indeed, one may explicit: L^lR^l is $R^\lrcorner$ with R^c replaced by 0, L^cR^c is $R^\lrcorner$ with R^l replaced by 0, and L^tR^t is $R^\lrcorner$ with R^l and R^c replaced by 0. This diagram is also a homotopy pushout diagram between objects which are cofibrant in the diagram-injective model structure on $R^\lrcorner$-modules. It follows that $R^\lrcorner$ is small in the homotopy category of $R^\lrcorner$-modules.

Since the original diagram of rings is a homotopy pullback, the unit of the adjunction is an equivalence for R, and we see that the generator R and the generator $R^\lrcorner$ correspond under the equivalence, as required in the hypothesis in Part (2) of Theorem 2.7.

Since R is cofibrant and generates R-mod, cellularization with respect to R has no effect on R-mod and we obtain the stated equivalence with the cellularization of $R^\lrcorner$-mod with respect to the diagram $R^\lrcorner$. □

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