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An algebraic model for free rational G-spectra for connected compact Lie groups G

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Abstract We show that the category of free rational G-spectra for a connected compact Lie group G is Quillen equivalent to the category of torsion differential graded modules over the polynomial ring $H^*(BG)$. The ingredients are the enriched Morita equivalences of Schwede and Shipley (Topology 42(1):103–153, 2003), the functors of Shipley (Am J Math 129:351–379, 2007) making rational spectra algebraic, Koszul duality and thick subcategory arguments based on the simplicity of the derived category of a polynomial ring.

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1 Introduction

For some time we have been working on the project of giving a concrete algebraic model for various categories of rational equivariant spectra. There are a number of sources of complexity: in the model theory, the structured spectra, the equivariant topology, the categorical apparatus and the algebraic models. We have found it helpful, both in our thinking and in communicating our results, to focus on the class of free spectra, essentially removing three sources of complication but still leaving a result of some interest.

The purpose of the present paper is to give a small, concrete and calculable model for free rational G-spectra for a connected compact Lie group G. The first attraction is that it is rather easy to describe both the homotopy category of free G-spectra and also the algebraic model. The homotopy category coincides with the category of rational cohomology theories on free G-spaces; better still, on free spaces an equivariant cohomology theory is the same as one in the naive sense (i.e., a contravariant functor on the category of free G-spaces satisfying the Eilenberg–Steenrod axioms and the wedge axiom). As for the algebraic model, we consider a suitable model category structure on differential graded (DG) torsion modules over the polynomial ring $H^*(BG) = H^*(BG;\mathbb{Q})$.

Theorem 1.1 *For any connected compact Lie group G, there is a zig-zag of Quillen equivalences*

$$\textit{free-}G\textit{-spectra}/\mathbb{Q} \simeq_Q \textit{tors-}H^*(BG)\textit{-mod}$$

of model categories. In particular their derived categories are equivalent

$$Ho(\textit{free-}G\textit{-spectra}/\mathbb{Q}) \simeq D(\textit{tors-}H^*(BG)\textit{-mod})$$

as triangulated categories.

The theorem has implications at several levels. First, it gives a purely algebraic model for free rational G-spectra; a property not enjoyed by all categories of spectra. Second, the algebraic model is closely related to the standard methods of equivariant topology, and can be viewed as a vindication of Borel's insights. Third, the model makes it routine to give calculations of maps between free G-spectra, for example by using the Adams spectral sequence (Theorem 6.1) below, and various structural questions (e.g., classification of thick and localizing subcategories) become accessible. Finally, the proof essentially proceeds by showing that certain categories are rigid, in the sense that in a suitable formal context, there is a unique category whose objects are modules with torsion homology over a commutative DGA R with $H^*(R)$ polynomial.

In Sect. 9 we also describe the counterparts of functors relating free G-spectra and free H-spectra where H is a connected subgroup of G. There is some interest to this, since if $i : H \longrightarrow G$ denotes the inclusion, the restriction functor i^* from G-spectra to H-spectra, has both a left adjoint i_* (induction) and a right adjoint $i_!$ (coinduction). Similarly, if $r : H^*(BG) \longrightarrow H^*(BH)$ is the induced map in cohomology, the restriction r^* from $H^*(BH)$-modules to $H^*(BG)$-modules has a left adjoint r_* (extension of scalars) and a right adjoint $r_!$ (coextension of scalars). In fact the middle functor i^* corresponds to the rightmost adjoint $r_!$ of the functors we have mentioned. It follows that i_* corresponds to r^*, but the right adjoint $i_!$ corresponds to a new functor; at the derived level $r_!$ is equivalent to a left adjoint $r_!'$, and $r_!'$ has a right adjoint $r^!$. In the end, just as $i_! \simeq \Sigma^{-c} i_*$, where $c = \dim(G/H)$, so too $r^! \simeq \Sigma^{-c} r^*$.

We are grateful to the referee for useful comments, especially for encouraging us to treat change of groups.

Convention 1.2 Certain conventions are in force throughout the paper. The most important is that *everything is rational*: henceforth all spectra and homology theories are rationalized without comment. For example, the category of rational G-spectra will now be denoted 'G-spectra'. The only exception to this rule is in the beginning of Sect. 3 where we introduce our model of (integral) free G-spectra. We also use the standard conventions that 'DG' abbreviates 'differential graded' and that 'subgroup' means 'closed subgroup'. We focus on homological (lower) degrees, with differentials reducing degrees; for clarity, cohomological (upper) degrees are called *codegrees* and are converted to degrees by negation in the usual way. Finally, we write $H^*(X)$ for the unreduced cohomology of a space X with rational coefficients.

2 Overview

2.1 Strategic summary

In this section we describe our overall strategy. Our task is to obtain a Quillen equivalence [20] between the category of rational free G-spectra and the category of DG torsion $H^*(BG)$-modules. In joining these two categories we have three main boundaries to cross: first, we have to pass from the realm of apparently unstructured homotopy theory to a category of modules, secondly we have to pass from a category of topological objects (spectra) to a category of algebraic objects (DG vector spaces), and finally we have to pass from modules over an arbitrary ring to modules over a commutative ring. These three steps could in principle be taken in any order, but we have found it convenient to begin by moving to modules over a ring spectrum, then to modules over a commutative ring spectrum and finally passing from modules over a commutative ring spectrum to modules over a commutative DGA. In the following paragraphs we explain the strategy, and point to the sections of the paper where the argument is given in detail. In Sect. 2.3 we then give a condensed outline of the argument. Various objects have counterparts in several of the worlds we pass through. As a notational cue we indicate this with a subscript, so that R_{top} is a topological object (a ring spectrum), R_t is an algebraic object (a DGA, but very large and poorly understood), and R_a is an algebraic object (another DGA, small and concrete).

2.2 Outline

In the homotopy category of free G-spectra, all spectra are constructed from free cells G_+, and rational G-spectra from rationalized free cells $\mathbb{Q}[G]$. There are a number of models for free rational G-spectra, but for definiteness we work with the usual model structure on orthogonal spectra [18], and localize it so that isomorphisms of non-equivariant rational homotopy become weak equivalences (see Sect. 3 for a fuller discussion).

The first step is an application of Morita theory, as given in the context of spectra by Schwede and Shipley [23]. Since all free rational G-spectra are constructed from the small object $\mathbb{Q}[G]$, and since the category is suitably enriched over spectra, the category is equivalent to the category of module spectra over the (derived) endomorphism ring spectrum

$$\mathcal{E}_{\mathrm{top}} = \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q}[G], \mathbb{Q}[G]) \simeq \mathbb{Q}[G].$$

Note that we have already reached a category of non-equivariant spectra: the category of free rational G-spectra is equivalent to the category of $\mathbb{Q}[G]$-modules in spectra (Proposition 3.3).

Having reached a module category (over the non-commutative ring $\mathbb{Q}[G]$), the next step is to move to a category of modules over a commutative ring. For this (see Sect. 4) we mimic the classical Koszul duality between modules over an exterior algebra (such as $H_*(G)$) and torsion modules over a polynomial ring (such as $H^*(BG)$) [3]. The equivalence takes a $\mathbb{Q}[G]$-module spectrum X to $X \wedge_{\mathbb{Q}[G]} \mathbb{Q}$, viewed as a left module over the (derived) endomorphism ring spectrum

$$R'_{\mathrm{top}} = \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q}, \mathbb{Q}).$$

The quasi-inverse to this takes an R'_{top}-module M to $\mathrm{Hom}_{R'_{\mathrm{top}}}(\mathbb{Q}, M)$. In the first instance this is a module over $\mathcal{E}'_{\mathrm{top}} = \mathrm{Hom}_{R'_{\mathrm{top}}}(\mathbb{Q}, \mathbb{Q})$, but note that multiplication by scalars defines the double centralizer map

$$\kappa : \mathcal{E}_{\mathrm{top}} \simeq \mathbb{Q}[G] \longrightarrow \mathrm{Hom}_{R'_{\mathrm{top}}}(\mathbb{Q}, \mathbb{Q}) = \mathcal{E}'_{\mathrm{top}}.$$

We may therefore obtain an $\mathcal{E}_{\mathrm{top}}$-module from an $\mathcal{E}'_{\mathrm{top}}$-module by restriction of scalars. We use an Adams spectral sequence argument to show that κ is a weak equivalence (Sect. 6), and it follows that the categories of $\mathcal{E}_{\mathrm{top}}$-modules and $\mathcal{E}'_{\mathrm{top}}$-modules are Quillen equivalent.

Now the Morita theory from [23] shows that the category of $\mathbb{Q}$-cellular R'_{top}-modules (i.e., those modules built from $\mathbb{Q}$ using coproducts and cofibre sequences) is equivalent to the category of $\mathcal{E}'_{\mathrm{top}}$-modules (see Sect. 5). The usual model of such cellular modules is obtained from a model for all modules by the the process of cellularization of model categories. Since we make extensive use of this, we provide an outline in Appendix A.

Unfortunately, R'_{top} is not actually commutative. Recalling that in the definition we take a convenient cofibrant and fibrant replacement $\mathbb{Q}[EG]$ of $\mathbb{Q}$, the ring operation on $R'_{\mathrm{top}} = \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q}[EG], \mathbb{Q}[EG])$ is composition. However an argument of Cartan from the algebraic case shows that it is quasi-isomorphic to the commutative ring spectrum $R_{\mathrm{top}} = \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q}[EG], \mathbb{Q})$, where the ring operation comes from that on $\mathbb{Q}$ and the diagonal of EG. Thus the category of free rational G-spectra is equivalent to the category of $\mathbb{Q}$-cellular modules over the commutative ring spectrum R_{top} (Proposition 5.2).

Finally, we are ready to move to algebra in Sect. 7. We have postponed this step as long as possible because it loses control of all but the most formal properties. Here we invoke the second author's equivalence between algebra spectra A_{top} over the rational Eilenberg–MacLane spectrum $H\mathbb{Q}$ and DGAs A_t over $\mathbb{Q}$. Under this equivalence, the corresponding module categories are Quillen equivalent. Because we are working rationally, if A_{top} is commutative, the associated DGA is weakly equivalent to a commutative DGA by [26, 1.2]. We apply this to the commutative ring spectrum R_{top} to obtain a DGA which is weakly equivalent to a commutative DGA, R_t. Since the equivalence preserves homotopy we find $H^*(R_t) = H^*(BG)$, and since this is a polynomial ring, we may construct a homology isomorphism $H^*(BG) \xrightarrow{\simeq} R_t$ by choosing cocycle representatives for the polynomial generators (Lemma 8.9). This shows the category of R_{top}-modules is equivalent to the category of modules over $H^*(BG)$. We want to take the cellularization of this equivalence, and it remains to show that the image of the R_{top}-module $\mathbb{Q}$ is the familiar $H^*(BG)$-module of the same name: for this we simply observe that all modules M with $H^*(M) = \mathbb{Q}$ are equivalent (Lemma 8.10). It follows that the category of free rational G-spectra is equivalent to the category of $\mathbb{Q}$-cellular $H^*(BG)$-modules (see Sect. 8.5 for a fuller discussion).

We have obtained a purely algebraic model for free rational G-spectra. It is then convenient to make it a little smaller and more concrete by showing that there is a model structure on DG torsion $H^*(BG)$-modules which is equivalent to $\mathbb{Q}$-cellular $H^*(BG)$-modules. It follows that the category of free rational G-spectra is equivalent to the category of torsion $H^*(BG)$-modules as stated in Theorem 1.1 (see Sect. 8 for a fuller discussion).

2.3 Diagrammatic summary

Slightly simplifying the above argument, we may summarize it by saying we have a string of Quillen equivalences

$$\text{free-}G\text{-spectra}/\mathbb{Q} \overset{1}{\simeq} \mathbb{Q}[G]\text{-mod} \overset{2}{\simeq} \mathbb{Q}\text{-cell-}R'_{\mathrm{top}}\text{-mod} \overset{3}{\simeq} \mathbb{Q}\text{-cell-}R_{\mathrm{top}}\text{-mod}$$
$$\overset{4}{\simeq} \mathbb{Q}\text{-cell-}R_t\text{-mod} \overset{5}{\simeq} \mathbb{Q}\text{-cell-}H^*(BG)\text{-mod} \overset{6}{\simeq} \text{tors-}H^*(BG)\text{-mod}$$

The first line takes place in topology, and the second in algebra. Equivalence (1) is a standard Morita equivalence. Equivalence (2) combines a Koszul Morita equivalence with a completeness statement and equivalence (3) uses Cartan's commutativity argument. Equivalence (4) applies the second author's algebraicization theorem to move from topology to algebra. Equivalence (5) uses the fact that $H^*(BG)$ is a polynomial ring and hence intrinsically formal together with a recognition theorem for the module $\mathbb{Q}$. Equivalence (6) applies some more elementary algebraic Quillen equivalences.

2.4 Discussion

In this paper we have presented an argument that is as algebraic as possible in the sense that after reducing to $\mathbb{Q}[G]$-modules, one can imagine making a similar argument (Koszul, completeness, Cartan, rigidity) in many different contexts with an algebraic character. It is intriguing that the present argument leaves the world of G-spectra so quickly. Generally this is bad strategy in equivariant matters, but its effectiveness here may be partly due to the fact we are dealing with free G-spectra. We use a different outline in [11] for torus-equivariant spectra which are not necessarily free.

One drawback of the present proof is that the equivalences have poor monoidal properties. To start with, it is not entirely clear how to give all of the categories symmetric monoidal products: for example some ingenuity is required for torsion $H^*(BG)$-modules, since there are not enough flat modules.

On the other hand, one can give a more topological argument. This relies on special properties of equivariant spectra, but it does have the advantage of having fewer steps and better monoidal properties. We plan to implement this in a future paper covering more general categories of spectra.

The restriction to connected groups is a significant simplification. To start with, the algebraic model for disconnected groups is more complicated. From the case of finite groups (where the category of free rational G-spectra is equivalent to the category of graded modules over the group ring [10]), we see that we can no longer expect a single indecomposable generator. Further complication is apparent for $O(2)$ (a split extension of $SO(2)$ by the group W of order 2), where the component group acts on $H^*(BSO(2))$ [1,2,7]. One expects more generally, that if G is the split extension of the identity component G_1 by the finite group W, the algebraic model consists of torsion modules over the twisted group ring $H^*(BG_1)[W]$. For non-split extensions like Spin(2), the situation is similar for homotopy categories, but the extension is encoded in the model category. The importance of the restriction to connected groups in the proof emerges in the convergence of the Adams spectral sequence.

3 The Morita equivalence for spectra

There are many possible models for equivariant G-spectra. The two most developed highly structured model categories for equivariant G-spectra are the category of G-equivariant EKMM S-modules [5,6] and the category of orthogonal G-spectra [18]. Due to technicalities in the proof of Proposition 3.1, we use orthogonal G-spectra here (see Remark 3.2 for a more detailed discussion). Recall that in this section we are not working rationally unless we explicitly say so. We need a model for rational free G-spectra, and this involves further choices. As we show below, one simple model for free G-spectra is the category of module spectra over $\mathbb{S}[G]$, the suspension spectrum of G with a disjoint basepoint added; analogously, a model for rational free G-spectra is the category of module spectra over the spectrum $H\mathbb{Q}[G] = H\mathbb{Q} \wedge G_+$. In this section we show that other, possibly more standard, models for free G-spectra and rational free G-spectra agree with these models.

3.1 Models for integral free G-spectra

Any model for the homotopy theory of free G-spectra will have a *compact* (*weak*) *generator* [23, 2.1.2] given by the homotopy type of the suspension spectrum of G_+, and we write $\mathbb{S}[G]$ for a representative of this spectrum. Morita theory [23, 3.9.3] (see also [24, 1.2]) then shows that, under certain hypotheses, a model category associated to the homotopy theory of free G-spectra is equivalent to the category of modules over the derived endomorphism ring spectrum of this generator. In the following, we show that for free G-spectra the derived endomorphism ring spectrum of $\mathbb{S}[G]$ is again weakly equivalent to $\mathbb{S}[G]$.

To be specific, here we model free G-spectra by using the category of orthogonal G-spectra and the underlying non-equivariant equivalences, that is, those equivalences detected by non-equivariant homotopy $\pi_*(-)$. In more detail, this model structure is given by cellularizing the model category of orthogonal G-spectra with respect to $\mathbb{S}[G]$ using Hirschhorn's machinery [12, 5.1.1] (see also Appendix A). Because we are working with free G-spectra, all universes from G-fixed to complete give equivalent models, but for definiteness we will use a complete G-universe. Here and for the rest of this section $\mathbb{S}[G]$ will denote the orthogonal G-spectrum taking each G-inner product space V in the universe to $\mathbb{S}[G](V) = S^V \wedge G_+$ where S^V is the one point compactification of V. Since G is compact and $[\mathbb{S}[G], -]_*$ detects the non-equivariant equivalences, $\mathbb{S}[G]$ is a compact generator for this model category (see also Proposition A.4). We will show that this category of free G-spectra satisfies the hypotheses of [23, 3.9.3], and identify the derived endomorphism ring of $\mathbb{S}[G]$ as $\mathbb{S}[G]$, to establish the following Morita equivalence.

Proposition 3.1 *The model category of free G-spectra described above is Quillen equivalent to the category of* (*right*) $\mathbb{S}[G]$*-modules in orthogonal spectra. In turn, this is Quillen equivalent to modules over the suspension spectrum of G_+ in any of the highly structured symmetric monoidal model categories of spectra.*

Proof The category of orthogonal G-spectra is compatibly tensored, cotensored and enriched over orthogonal spectra [18, II.3.2, III.7.5]. Specifically, given two orthogonal G-spectra X, Y the enrichment over orthogonal G-spectra is denoted $F_{S_G}(X, Y)$, and the enrichment over orthogonal spectra is given by taking G-fixed points. Since $\mathbb{S}[G]$ is cofibrant and a compact generator of free G-spectra, it follows from [24, 1.2] that the model category of free G-spectra is Quillen equivalent to modules over the endomorphism ring spectrum $\mathrm{End}_G(f\mathbb{S}[G]) = F_{S_G}(f\mathbb{S}[G], f\mathbb{S}[G])^G$ where f is any fibrant replacement functor in orthogonal G-spectra.

Since $\mathbb{S}[G]$ is a monoid and fibrant monoids are fibrant as underlying spectra, we take f to be the fibrant replacement in the category of monoids in orthogonal G-spectra.

We make use of the equivalences of ring spectra

$$\mathbb{S}[G] \simeq (\mathbb{S}[G])^{\mathrm{op}} \simeq F_{S_G}(\mathbb{S}[G], \mathbb{S}[G])$$

where the first equivalence is the inverse map. For the second equivalence, note that the change of groups equivalence holds in the strong sense that since $\mathbb{S}[G] = S_G \wedge G_+$, for any G-space X the function spectrum $F_{S_G}(S_G \wedge G_+, X)$ is given by the levelwise based function space $\mathrm{map}(G_+, X)$ whose G-fixed point space is just X.

Restriction and extension of scalars over a weak equivalence of orthogonal ring spectra induce a Quillen equivalence between the categories of modules by [24, 7.2]; see also [19]. Applying this several times we see that once we show that $F_{S_G}(\mathbb{S}[G], \mathbb{S}[G])$ and $F_{S_G}(f\mathbb{S}[G], f\mathbb{S}[G])$ are weakly equivalent as ring spectra, the model categories of module spectra over $\mathrm{End}_G(f\mathbb{S}[G])$ and module spectra over $\mathbb{S}[G]$ are Quillen equivalent.

We first show that we have a zig-zag of weak equivalences of module spectra

$$F_{S_G}(\mathbb{S}[G], \mathbb{S}[G]) \xrightarrow{\simeq} F_{S_G}(\mathbb{S}[G], f\mathbb{S}[G]) \overset{\simeq}{\leftarrow} F_{S_G}(f\mathbb{S}[G], f\mathbb{S}[G]).$$

The first of these is a weak equivalence since it is isomorphic to $\mathbb{S}[G] \longrightarrow f\mathbb{S}[G]$. The second map is a trivial fibration since the enrichment over orthogonal spectra satisfies the analogue of SM7 [19, 12.6] and in the first variable we have a trivial cofibration and the second variable is a fibrant object. Since the fibrant replacement functor here is in the category of monoids, we must use the fact that trivial cofibrations of monoids forget to trivial cofibrations by Schwede and Shipley [22].

This zig-zag of module level equivalences is actually a quasi-equivalence [23, A.2.2]. Namely, these maps make $F_{S_G}(\mathbb{S}[G], f\mathbb{S}[G])$ a bimodule over $\mathrm{End}(\mathbb{S}[G])$ and $\mathrm{End}(f\mathbb{S}[G])$ and the maps are given by right and left multiplication with the fibrant replacement map $\mathbb{S}[G] \to f\mathbb{S}[G]$ (thought of as an "element" of the bimodule.) A zig-zag of equivalences of ring spectra can then be created from this quasi-equivalence. Construct $\mathcal{P}$ as the pull back in orthogonal spectra of the quasi-equivalence diagram.

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\simeq} & F_{S_G}(f\mathbb{S}[G], f\mathbb{S}[G]) \\ \downarrow\simeq & & \downarrow\simeq \\ F_{S_G}(\mathbb{S}[G], \mathbb{S}[G]) & \xrightarrow{\simeq} & F_{S_G}(\mathbb{S}[G], f\mathbb{S}[G]) \end{array}$$

Then $\mathcal{P}$ has the unique structure as a orthogonal ring spectrum such that the maps $\mathcal{P} \to F_{S_G}(\mathbb{S}[G], \mathbb{S}[G])$ and $\mathcal{P} \to F_{S_G}(f\mathbb{S}[G], f\mathbb{S}[G])$ are homomorphisms of ring G-spectra. Since the right displayed map in the quasi-equivalence is a trivial fibration, the pull back left map is also. Since the bottom map is also a weak equivalence, the top map out of the pull-back is a weak equivalence as well by the two out of three property.

According to [24, Cor. 1.2], the Morita theorem from [23, 3.9.3] can be restated to give an equivalence with modules over endomorphism ring spectrum in any of the highly structured symmetric monoidal model categories of spectra. It is also easy to see that the comparison functors between these various models preserve suspension spectra. So, since $\mathbb{S}[G]$ is a suspension spectrum, the comparison theorems of [19,21] show that one can model free G-spectra in any of these settings as the module spectra over the suspension spectrum of G_+. □

Remark 3.2 To model free G-spectra directly using [6], one would again need to identify the derived endomorphism ring of the suspension spectrum of G_+. Since all spectra are fibrant in [6], one would then consider the endomorphism ring of a cofibrant replacement. To mimic the proof above one would need this to be a cofibrant replacement as a ring spectrum at one point and a cofibrant replacement as a module spectrum at another. This technical complication persuaded us to use orthogonal spectra instead.

3.2 Models for rational free G-spectra

We now need a model for *rational* free G-spectra. One can localize any of the models for free G-spectra with respect to rationalized non-equivariant equivalences, given by $\pi_*(-) \otimes \mathbb{Q}$. The statements in this subsection hold in any of the highly structure monoidal model categories of spectra; to be specific though, we use orthogonal spectra. Very similar proofs hold in other models. In particular, let $L_{\mathbb{Q}}(\mathbb{S}[G]\text{-mod})$ denote the localized model structure given by [12, 4.1.1]. As above, one can also model this using spectral algebra; we will show that another model for rational free G-spectra is given by $H\mathbb{Q}[G]$-module spectra where $H\mathbb{Q}[G]$ denotes the spectrum $H\mathbb{Q} \wedge \mathbb{S}[G] \cong H\mathbb{Q} \wedge G_+$.

Proposition 3.3 *The localization of free G-spectra with respect to rational equivalences, $L_{\mathbb{Q}}(\mathbb{S}[G]$-mod), is Quillen equivalent to the category of $H\mathbb{Q}[G]$-module spectra. Thus $H\mathbb{Q}[G]$-module spectra is a model of rational free G-spectra.*

Proof Consider the functor $H\mathbb{Q} \wedge (-)$ from $L_{\mathbb{Q}}(\mathbb{S}[G]\text{-mod})$ to $H\mathbb{Q}[G]$-modules. Since $\mathbb{Q}$ is flat over π_*^S, one can use the Tor spectral sequence from [6, IV.4.1] to show that the derived homotopy of $H\mathbb{Q} \wedge X$ is isomorphic to the rationalized derived homotopy of X; that is, $\pi_*(f(H\mathbb{Q} \wedge X)) \cong \pi_*(fX) \otimes \mathbb{Q}$ for any cofibrant $\mathbb{S}[G]$-module spectrum X. Thus, $H\mathbb{Q} \wedge -$ preserves weak equivalences. Since cofibrations in $L_{\mathbb{Q}}(\mathbb{S}[G]\text{-mod})$ agree with those in $\mathbb{S}[G]$-mod, $H\mathbb{Q} \wedge -$ also preserves cofibrations and is thus a left Quillen functor. Denote its right adjoint by U.

To show that these functors induce a Quillen equivalence, let X be cofibrant in $L_{\mathbb{Q}}(\mathcal{S}_\Sigma[G]\text{-mod})$ and Y be fibrant in $H\mathbb{Q}[G]$-mod. The map $H\mathbb{Q} \wedge X \to Y$ is a weak equivalence if and only if $\pi_* f(H\mathbb{Q} \wedge X) \cong \pi_* Y$. By the above calculation for the source, this is true if and only if $\pi_*(fX) \otimes \mathbb{Q} \cong \pi_*(Y) \otimes \mathbb{Q}$ since Y is a $H\mathbb{Q}[G]$-module and its homotopy groups are rational. But this holds if and only if the map $X \to UY$ is a rational weak equivalence. □

Note, the convention that everything is rational is back in place after this section. In particular, $H\mathbb{Q}[G]$ is henceforth denoted by $\mathbb{S}[G]$; it was denoted $\mathbb{Q}[G]$ in Sect. 2.

4 An introduction to Koszul dualities

In this section we describe a general (implicitly enriched) strategy for proving a Koszul duality result,

$$\text{mod-}\mathcal{E} \simeq k\text{-cell-}R\text{-mod},$$

and illustrate it in the motivating algebraic example (explicitly enriched over chain complexes). We then give a proof of the topological case (explicitly enriched over spectra) in Sect. 5.

To describe the general strategy, we start from mod-$\mathcal{E}$, an enriched model category of modules over a ring $\mathcal{E}$. We consider a *bifibrant* (that is, cofibrant and fibrant) $\mathcal{E}$-module k and its enriched endomorphism ring, $R' = \mathrm{Hom}_{\mathcal{E}}(k, k)$ (we will later use the simpler notation R

for an equivalent *commutative* ring). Evaluation then gives k a left R'-module structure, and we obtain a functor

$$E' : R'\text{-mod} \longrightarrow \text{mod-}\mathcal{E}'$$

with $\mathcal{E}' = \mathrm{Hom}_{R'}(k, k)$ and

$$E'(M) = \mathrm{Hom}_{R'}(k, M).$$

This takes values in mod-$\mathcal{E}'$ since $\mathrm{Hom}_{R'}(k, M)$ is a right $\mathcal{E}'$-module by composition. By construction k is a generator of k-cell-R'-mod, when k is small in the homotopy category [23, 2.1.2] as an R'-module, so Morita theory applies here to give a Quillen equivalence between k-cell-R'-mod and mod-$\mathcal{E}'$. This statement is proved as part of Theorem 5.4 below.

We may then restrict along the double centralizer map [4]

$$\kappa : \mathcal{E} \longrightarrow \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{E}}(k,k)}(k, k) = \mathrm{Hom}_{R'}(k, k) = \mathcal{E}'$$

adjoint to the action map. Composing then gives a functor

$$E : R'\text{-mod} \longrightarrow \text{mod-}\mathcal{E}$$

where $E = \kappa^* E'$, so that $E(M) = \mathrm{Hom}_{R'}(k, M)$, now thought of as an $\mathcal{E}$-module. Provided the map κ is a weak equivalence we combine this with the Morita equivalence between k-cell-R'-mod and mod-$\mathcal{E}'$ mentioned above to prove in Theorem 5.4 below that we have a Quillen equivalence

$$k\text{-cell-}R'\text{-mod} \simeq \text{mod-}\mathcal{E}.$$

To motivate the terminology and display the obstacles in a familiar context, we describe the classical example in our terms. A subscript a is added here to specify that this is the algebraic case.

Example 4.1 The algebraic example works with the ambient category of DG k modules (i.e., chain complexes) for a field k and

$$\mathcal{E} = \mathcal{E}_a = H_*(G),$$

an exterior algebra on a finite dimensional vector space in odd degrees, with trivial differential. Up to equivalence (see more detail in the next paragraph) we find

$$R = R_a \simeq H^*(BG),$$

a symmetric algebra on a finite dimensional vector space in even codegrees. The conclusion is the statement

$$\text{mod-}H_*(G) \simeq k\text{-cell-}H^*(BG)\text{-mod}.$$

This is a Koszul duality, since we will describe in Sect. 8 below how k-cell-$H^*(BG)$-mod is a particular model for the category of torsion modules over the polynomial ring $H^*(BG)$.

It may be helpful to be a bit more precise, to highlight some of the technical issues we need to tackle. Choosing the projective model structure on the category of $\mathcal{E}_a$-modules, all objects are fibrant and therefore a bifibrant version of k is given by a projective resolution k_c (in this case we could be completely explicit). Then we have $R'_a = \mathrm{Hom}_{\mathcal{E}_a}(k_c, k_c)$. We then argue that we may choose k_c so that it has a cocommutative diagonal map and hence a Cartan commutativity argument (see Proposition 5.2) gives a ring map to the commutative DGA $\mathrm{Hom}_{\mathcal{E}_a}(k_c, k)$. It is formal that this map is a weak equivalence; thus R'_a is equivalent to $R_a = \mathrm{Hom}_{\mathcal{E}_a}(k_c, k)$. Since its homology is the polynomial ring $H^*(BG)$, we may pick representative cycles in even codegrees to give a homology isomorphism

$$H^*(BG) \longrightarrow \mathrm{Hom}_{\mathcal{E}_a}(k_c, k).$$

Note, this implicitly uses the intrinsic formality of a polynomial ring on even degree generators (Lemma 8.9).

Returning to the main argument, k_c is a left R'_a-module by construction. To form $\mathcal{E}'_a$, we replace k_c by a version bifibrant in R'_a-modules; again, one could be completely explicit. To see that the comparison map κ is a homology isomorphism, we use the classical calculation $\mathrm{Ext}^{*,*}_{H^*(BG)}(k, k) \cong H_*(G)$, which could be viewed as a collapsed Eilenberg–Moore spectral sequence.

5 The topological Koszul duality

In this section we establish the Quillen equivalence

$$\text{mod-}\mathcal{E} \simeq k\text{-cell-}R\text{-mod}$$

in the topological setting (enriched over spectra) using the strategy of Sect. 4.

Before turning to the proof, we describe the topological example, following the pattern of Example 4.1. The topological example works best with the ambient category of EKMM spectra [6] because all spectra are fibrant in this model, so we work with EKMM spectra throughout this section; by Proposition 3.1 free rational G-spectra are modeled by $\mathbb{S}[G]$-module spectra in EKMM spectra. We then use the comparison functors of [21] to pass to symmetric spectra on the way to shifting to algebra in Sect. 7. Note as well that these comparison functors take the symmetric spectrum $\mathcal{S}_\Sigma[G]$ [15] to the EKMM suspension spectrum of G_+, $\mathbb{S}[G]$.

Example 5.1 We take

$$\mathcal{E} = \mathcal{E}_{\mathrm{top}} = \mathbb{S}[G],$$

the suspension spectrum of G_+ from [6]. The cofibrant objects in $\mathcal{E}_{\mathrm{top}}$-modules are those built from $\mathbb{S}[G]$ (i.e., retracts of $\mathbb{S}[G]$-cell objects) and weak equivalences are created in spectra ($\mathbb{S}$-modules). The sphere spectrum $\mathbb{S}$ is an $\mathbb{S}[G]$-module by using the augmentation $\mathbb{S}[G] \longrightarrow \mathbb{S}[*] = \mathbb{S}$. Since $\mathbb{S}$ is not cofibrant as an $\mathbb{S}[G]$-module, we consider a cofibrant replacement

$$\mathbb{S}_c = \mathbb{S}[EG] = \mathbb{S} \wedge EG_+.$$

We then take $k_{\mathrm{top}} = \mathbb{S}_{cf} = \mathbb{S}[EG]$ since it is cofibrant and fibrant (*bifibrant*) and weakly equivalent to $\mathbb{S}$.

Now, using the internal enrichment of EKMM spectra we define

$$R'_{\mathrm{top}} = F_{\mathcal{E}_{\mathrm{top}}}(k_{\mathrm{top}}, k_{\mathrm{top}}) = F_{\mathbb{S}[G]}(\mathbb{S}_{cf}, \mathbb{S}_{cf}).$$

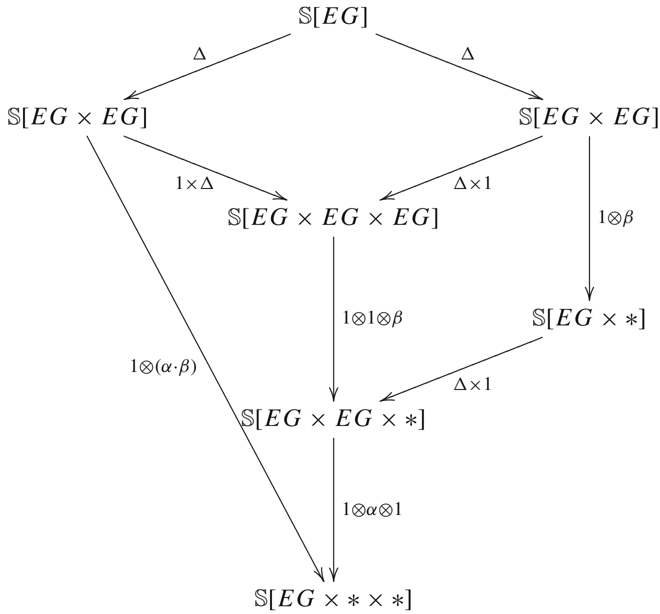
It is central to our formality argument that this is equivalent to a commutative ring spectrum.

Proposition 5.2 *The ring spectrum* $R'_{top} = F_{\mathbb{S}[G]}(\mathbb{S}[EG], \mathbb{S}[EG])$ *(with ring operation given by composition) is equivalent to the commutative ring spectrum* $R_{top} = F_{\mathbb{S}[G]}(\mathbb{S}[EG], \mathbb{S})$ *(with ring operation coming from the diagonal map of the space* EG*).*

Proof First, we define the map:

$$\begin{aligned}\theta : R_{\text{top}} = F_{\mathbb{S}[G]}(\mathbb{S}[EG], \mathbb{S}) &\xrightarrow{\wedge_{\mathbb{S}}\mathbb{S}[EG]} F_{\mathbb{S}[G]}(\mathbb{S}[EG] \wedge_{\mathbb{S}} \mathbb{S}[EG], \mathbb{S} \wedge_{\mathbb{S}} \mathbb{S}[EG]) \\ &= F_{\mathbb{S}[G]}(\mathbb{S}[EG \times EG], \mathbb{S}[EG]) \xrightarrow{\Delta^*} F_{\mathbb{S}[G]}(\mathbb{S}[EG], \mathbb{S}[EG]) = R'_{\text{top}}.\end{aligned}$$

To see this is a ring map, we use the following commutative diagram



Following down the left hand side we obtain $\theta(\alpha \cdot \beta)$, whereas the right hand side gives $\theta(\alpha) \circ \theta(\beta)$ as required. $\square$

Remark 5.3 Note also that we have an equivalence of ring spectra

$$R_{\text{top}} = F_{\mathbb{S}[G]}(\mathbb{S}[EG], \mathbb{S}) \simeq F_{\mathbb{S}}(\mathbb{S}[BG], \mathbb{S}) = \overline{R}_{\text{top}}$$

if one wants to minimise the appearance of equivariant objects.

Noting that $\mathbb{S}[G]$ is analogous to chains on G and the final term to cochains on BG (both with coefficients in $\mathbb{S}$), the equivalence $R_{\text{top}} \simeq \overline{R}_{\text{top}}$ is an analogue of the Rothenberg–Steenrod theorem.

Finally, we consider the map

$$\kappa_{\text{top}} : \mathcal{E}_{\text{top}} = \mathbb{S}[G] \xrightarrow{\simeq} F_{R'_{\text{top}}}(\mathbb{S}_{cf}, \mathbb{S}_{cf}) = \mathcal{E}'_{\text{top}}.$$

Using crude versions of the arguments constructing the Adams spectral sequence of Sect. 6, we show that this is an equivalence in Lemma 6.7.

The next result holds in great generality, but here we only consider the two relevant cases of enrichment over chain complexes and over spectra. Let $\mathcal{E}$, $\mathcal{E}'$, R' and k be as described at the start of Sect. 4.

Theorem 5.4 *In the algebraic and topological settings considered here, if k is a bifibrant and compact R'-module and the double centralizer map $\kappa : \mathcal{E} \longrightarrow \mathcal{E}'$ is a weak equivalence, then there is a Quillen equivalence*

$$k\text{-}cell\text{-}R'\text{-}mod \simeq mod\text{-}\mathcal{E}.$$

In both the algebraic case (Example 4.1) and the spectral case (Example 5.1) we have verified the hypotheses of this theorem. Thus we have given a proof of the usual Koszul duality statement, and of the topological version we require. Since R'_{top} and R_{top} are weakly equivalent by Proposition 5.2, their module categories are Quillen equivalent by [22, 4.3].

Corollary 5.5 *There are Quillen equivalences*

$$mod\text{-}\mathcal{E}_{top} \simeq k\text{-}cell\text{-}R'_{top}\text{-}mod \simeq k\text{-}cell\text{-}R_{top}\text{-}mod.$$

Proof of 5.4 This Quillen equivalence follows in two steps. The simplest step is that restriction and extension of scalars across a weak equivalence induces a Quillen equivalence between the categories of modules

$$\text{mod-}\mathcal{E} \simeq_Q \text{mod-}\mathcal{E}'.$$

This holds by [22, 4.3].

The other step is to establish the Quillen equivalence with right adjoint

$$E' : k\text{-cell-}R'\text{-mod} \to \text{mod-}\mathcal{E}'.$$

For the model structure on k-cell-R'-mod, we consider the cellularization of R'-mod with respect to the object k, see Proposition A.2. Here the fibrations are again the underlying fibrations in R'-mod but the weak equivalences are the maps $g : M \to N$ such that $g_* : \text{Hom}_{R'}(k, fM) \longrightarrow \text{Hom}_{R'}(k, fN)$ are weak equivalences (of $\mathbb{S}$-modules or chain complexes) where f is a fibrant replacement functor in R'-mod. The cofibrant objects here are the k-cellular objects; those built from k by disk-sphere pairs. In particular, since k was bifibrant in R'-mod it is also bifibrant in k-cell-R'-mod.

Since k is compact and it detects weak equivalences in k-cell-R'-mod, it is a compact (weak) generator by [23, 2.2.1]. Since k is cofibrant and fibrant in k-cell-R'-mod as well as R'-mod, its derived endomorphism ring is $\mathcal{E}' = \text{Hom}_{R'}(k, k)$. It follows by [24, 1.2] for spectra or by Shipley [25] for DG modules that E' is the right adjoint in a Quillen equivalence. □

6 The Adams spectral sequence

The following Adams spectral sequence is both an effective calculational tool, and a quantitative version of a double-centralizer statement. It states that for free G-spectra, the stable equivariant homotopy groups $\pi_*^G(X) = [S^0, X]_*^G$ give a complete and effective invariant.

Theorem 6.1 *Suppose G is a connected compact Lie group. For any free rational G-spectra X and Y there is a natural Adams spectral sequence*

$$\text{Ext}^{*,*}_{H^*(BG)}(\pi_*^G(X), \pi_*^G(Y)) \Rightarrow [X, Y]_*^G.$$

It is a finite spectral sequence concentrated in rows 0 *to* r *and strongly convergent for all* X *and* Y *where* $r = \text{rank}(G)$.

This is useful for calculation, but we only need the less explicit version in Sect. 6.3 for the proof of our main theorem.

6.1 Standard operating procedure

We apply the usual method for constructing an Adams spectral sequence based on a homology theory H_* on a category $\mathbb{C}$ with values in an abelian category $\mathcal{A}$ (in our case $\mathbb{C}$ is the category of free rational G-spectra, $\mathcal{A}$ is the category of torsion $H^*(BG)$-modules and $H_* = \pi_*^G$). It may be helpful to summarize the process: to construct an Adams spectral sequence for calculating $[X, Y]$ we proceed as follows.

Step 0 Take an injective resolution

$$0 \longrightarrow H_*(Y) \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

in $\mathcal{A}$.

Step 1 Show that enough injectives I of $\mathcal{A}$ (including the I_j) can be realized by objects $\mathbb{I}$ of $\mathbb{C}$ in the sense that $H_*(\mathbb{I}) \cong I$.

Step 2 Show that the injective case of the spectral sequence is correct in that homology gives an isomorphism

$$[X, Y] \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(H_*(X), H_*(Y))$$

if Y is one of the injectives $\mathbb{I}$ used in Step 1.

Step 3 Now construct the Adams tower

$$\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
Y_2 & \longrightarrow & \Sigma^{-2}\mathbb{I}_2 \\
\downarrow & & \\
Y_1 & \longrightarrow & \Sigma^{-1}\mathbb{I}_1 \\
\downarrow & & \\
Y_0 & \longrightarrow & \Sigma^{0}\mathbb{I}_0
\end{array}$$

over $Y = Y_0$ from the resolution. This is a formality from Step 2. We work up the tower, at each stage defining Y_{j+1} to be the fibre of $Y_j \longrightarrow \Sigma^{-j}\mathbb{I}_j$, and noting that $H_*(Y_{j+1})$ is the $(j+1)$st syzygy of $H_*(Y)$.

Step 4 Apply $[X, \cdot]$ to the tower. By the injective case (Step 2), we identify the E_1 term with the complex $\text{Hom}^*_{\mathcal{A}}(H_*(X), I_\bullet)$ and the E_2 term with $\text{Ext}^{*,*}_{\mathcal{A}}(H_*(X), H_*(Y))$.

Step 5a If the injective resolution is infinite, the first step of convergence is to show that $H_*(\underset{\leftarrow\, j}{\text{holim}}\, Y_j)$ is calculated using a Milnor exact sequence from the inverse system

$\{H_*(Y_j)\}_j$, and hence that $H_*(\operatorname{holim}_{\leftarrow j} Y_j) = 0$. If the injective resolution is finite, this is automatic.

Step 5b Deduce convergence from Step 5a by showing $\operatorname{holim}_{\leftarrow j} Y_j \simeq *$. In other words we must show that $H_*(\cdot)$ detects isomorphisms in the sense that $H_*(Z) = 0$ implies $Z \simeq *$. In general, one needs to require that $\mathbb{C}$ is a category of appropriately complete objects for this to be true. This establishes conditional convergence. If $\mathcal{A}$ has finite injective dimension, finite convergence is then immediate.

6.2 Implementation

In our case, Step 0 is a well known piece of algebra: torsion modules over the polynomial ring $H^*(BG)$ admit injective resolutions consisting of sums of copies of the basic injective $I = H_*(BG)$. Step 1 now follows from a familiar calculation.

Lemma 6.2 *The injectives are realized in the sense that*

$$\pi_*^G(EG_+) = \Sigma^d I,$$

where d is the dimension of G.

Proof We calculate

$$\pi_*^G(EG_+) \cong \pi_*(BG^{ad}) \cong H_*(BG^{ad}) \cong \Sigma^d H_*(BG) = \Sigma^d I,$$

where the third isomorphism used the fact that since G is connected, the adjoint representation is orientable. □

Steps 3 and 4 are formalities, and Step 5a is automatic since the category of modules over the polynomial ring $H^*(BG)$ is of finite injective dimension. This leaves Steps 2 and 5b.

For Step 2 we prove the injective case of the Adams spectral sequence.

Lemma 6.3 *Suppose Y is any wedge of suspensions of EG_+. For any free G-spectrum X, application of π_*^G induces an isomorphism*

$$[X, Y]^G \xrightarrow{\cong} \operatorname{Hom}_{H^*(BG)}(\pi_*^G(X), \pi_*^G(Y)).$$

Proof Since $\pi_*^G(Y)$ is injective, both sides are cohomology theories in X, so it suffices to establish the special case $X = G_+$. Since G_+ is small and $\pi_*^G(G_+)$ is finitely generated, it suffices to deal with the special case $Y = EG_+$.

In other words, we must show that

$$\pi_*^G : [G_+, EG_+]^G \xrightarrow{\cong} \operatorname{Hom}_{H^*(BG)}(\pi_*^G(G_+), \pi_*^G(EG_+))$$

is an isomorphism. Now both sides consist of a single copy of $\mathbb{Q}$ in degree 0, so it suffices to show the map is non-trivial. This is clear since a non-trivial G-map $f : G_+ \longrightarrow EG_+$ gives a map $f/G : S^0 = (G_+)/G \longrightarrow (EG_+)/G = BG_+$ which is non-trivial in reduced H_0. □

Finally, for Step 5b we prove the universal Whitehead theorem. This is where the connectedness of G is used.

Lemma 6.4 *If G is connected, then the functor π_*^G detects isomorphisms in the sense that if $f : Y \longrightarrow Z$ is a map of free G-spectra inducing an isomorphism $f_* : \pi_*^G(Y) \longrightarrow \pi_*^G(Z)$ then f is an equivalence.*

Proof Since π^G_* is exact, it suffices to prove that if $\pi^G_*(X)=0$ then $\pi_*(X)=\pi^G_*(G_+\wedge X)=0$.

For a general connected group we may argue as follows provided we use the fact that $R_{\mathrm{top}} \simeq F(EG_+, \mathbb{S})$ is a commutative ring spectrum. Indeed, if $H^*(BG) = \mathbb{Q}[x_1, \dots, x_r]$ we may view each generator x_i as a map of R_{top}-modules. As such we may form the Koszul complex

$$K_{R_{\mathrm{top}}}(x_1, \dots, x_i) = \mathrm{fibre}(R_{\mathrm{top}} \xrightarrow{x_1} R_{\mathrm{top}}) \wedge_{R_{\mathrm{top}}} \dots \wedge_{R_{\mathrm{top}}} \mathrm{fibre}(R_{\mathrm{top}} \xrightarrow{x_i} R_{\mathrm{top}}),$$

and we find

$$K_{R_{\mathrm{top}}}(x_1, \dots, x_r) \wedge EG_+ \simeq G_+.$$

Now we argue by induction on i that

$$\pi^G_*(K_{R_{\mathrm{top}}}(x_1, \dots, x_i) \wedge X) = 0.$$

The case $i = 0$ is the hypothesis and the case $i = r$ is the required conclusion. However the fibre sequence

$$K_{R_{\mathrm{top}}}(x_1, \dots, x_{i+1}) \longrightarrow K_{R_{\mathrm{top}}}(x_1, \dots, x_i) \xrightarrow{x_i} K_{R_{\mathrm{top}}}(x_1, \dots, x_i)$$

shows that the ith case implies the $(i + 1)$st. □

Remark 6.5 (i) If G acts freely on a product of spheres we may argue as in [9]; this is essentially the same as the present argument, but spheres and Euler classes realize the homotopy generators and avoid the need to use a triangulated category of R_{top}-modules.

(ii) There are other ways to make this argument with less technology. Note that since we are working rationally and X is free $\pi^G_*(X) = H_*(X/G)$ and $\pi_*(X) = H_*(X)$. Suppose first that X is a 1-connected space. We may consider the Serre Spectral Sequence for the fibre sequence $G \longrightarrow X \longrightarrow X/G$. Since G is connected X/G is simply connected and the spectral sequence

$$H_*(X/G; H_*(G)) \Rightarrow H_*(X)$$

is untwisted. By hypothesis it has zero E^2-term and hence $H_*(X) = 0$. Now for a general spectrum we may express X as a filtered colimit of finite spectra. Each of these finite spectra is a suspension of a space, and since the spaces are free, the suspensions may be taken to be by trivial representations. Accordingly there is a Serre Spectral Sequence for each of the finite spectra, and by passing to direct limits, there is a similar spectral sequence for X itself, and we may make the same argument.

6.3 A geometric counterpart of the Adams spectral sequence

The completion result that we need is really a non-calculational version of part of the construction of the Adams spectral sequence. Despite its non-calculational nature, it is convenient to use special cases of the arguments we have used earlier in the section, which is why we have placed it here.

Proposition 6.6 *The enriched functor* $\mathbb{S}[EG] \wedge_{\mathbb{S}[G]} (\cdot)$ *gives a natural equivalence*

$$\nu_{X,Y} : F_{\mathbb{S}[G]}(X, Y) \longrightarrow F_{R_{top}}(\mathbb{S}[EG] \wedge_{\mathbb{S}[G]} X, \mathbb{S}[EG] \wedge_{\mathbb{S}[G]} Y)$$

of spectra for all $\mathbb{S}[G]$*-modules* X *and* Y.

Proof First note that, for each fixed Y, the class of X for which $\nu_{X,Y}$ is an equivalence is closed under completion of triangles and arbitrary wedges. Next note that $\nu_{X,Y}$ is an equivalence when $X = \mathbb{S}[G]$ and $Y = \mathbb{S}[EG]$ by the argument of Lemma 6.3. Since $\mathbb{S}[G]$ is small, it follows that ν is an equivalence for $X = \mathbb{S}[G]$ when Y is an arbitrary wedge of suspensions of modules $\mathbb{S}[EG]$. It follows that for these Y, the map ν is an equivalence for arbitrary X.

Next, note that, for each fixed X, the class of Y for which $\nu_{X,Y}$ is an equivalence is closed under completion of triangles. The construction of Adams resolutions shows that an arbitrary $\mathbb{S}[G]$-module is built from wedges of suspensions of $\mathbb{S}[EG]$ using finitely many triangles, so it follows that ν is an equivalence for arbitrary X and Y as claimed. □

Finally, note that if we take $X = Y = \mathbb{S}[G]$ we obtain the double centralizer map κ_{top}, giving the required corollary.

Corollary 6.7 *The map*

$$\kappa_{top} : \mathbb{S}[G] = F_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G]) \xrightarrow{\simeq} F_{R_{top}}(\mathbb{S}, \mathbb{S}) = \mathcal{E}'_{top}$$

is an equivalence.

Remark 6.8 The proof of the corollary alone can be considerably simplified, since we may fix $X = \mathbb{S}[G]$, and we may show that $Y = \mathbb{S}[G]$ is built from $Y = \mathbb{S}[EG]$ by giving an explicit Adams resolution of G_+ modelled on the dual of the Koszul complex as in [9, Section 12].

7 From spectra to chain complexes

In this section we show that the topological model category k-cell-R_{top}-mod is Quillen equivalent to the algebraic model category $\mathbb{Q}$-cell-R_t-mod. We first (implicitly) replace R_{top} by the associated symmetric ring spectrum using the lax symmetric monoidal comparison functors in [21] or [24, 1.2]; we do not change the notation for R_{top} here to emphasize the simplicity of changing models of spectra. To move from $H\mathbb{Q}$-algebras to rational DGAs we use the functor Θ developed in [26]; explicitly, in the notation of [26] we have $\Theta = Dc(\phi^* N)Zc$.

Proposition 7.1 *The DGA ΘR_{top} is weakly equivalent to a commutative rational DG algebra R_t.*

Proof This follows from [26, 1.2]. □

By [26, 2.15] and [22, 4.3], we then have the following.

Corollary 7.2 *There are Quillen equivalences between DG R_t-modules and R_{top}-module spectra.*

$$R_{top}\text{-}mod \simeq_Q R_t\text{-}mod.$$

We now apply Proposition A.5 to the zig-zag of Quillen equivalences between R_{top}-module spectra and DG R_t-modules from Corollary 7.2. We want to find the algebraic equivalent of the k-cellularization of R_{top}-mod. Define k_t to be the image of k under the various derived Quillen functors involved in the zig-zag of equivalences between R_{top} and R_t modules. This will involve some extension or restriction functors across weak equivalences and functors from [26].

Corollary 7.3 *The k-cellularization of R_{top}-module spectra is Quillen equivalent to the k_t-cellularization of R_t.*

$$k\text{-}cell\text{-}R_{top}\text{-}mod \simeq_Q k_t\text{-}cell\text{-}R_t\text{-}mod$$

8 Models of the category of torsion $H^*(BG)$-modules

In this section we consider the category of R-modules, where R is a commutative DGA over k with $H^*(R) = H^*(BG)$ (for example $R = H^*(BG)$ or $R = R_t$ with $k = \mathbb{Q}$). More precisely, we need to consider models for the category of R-modules with torsion homology. In general we can obtain a model by cellularizing a model of all R-modules with respect to the residue field k; if R has zero differential there is an alternative model with underlying category consisting of the DG torsion modules. We will show that the various models are Quillen equivalent.

8.1 An algebraic template

We begin with an overview of a simple and explicit example. We give detailed proofs in Sects. 8.3 and 8.5 below after adapting it to the general case.

Consider the commutative ring $R = k[c]$ with c of degree -2 (this corresponds to the case when G is the circle group). We may consider the abelian category R-mod of DG R-modules, and the abelian subcategory tors-R-mod of DG R-modules M with $M[1/c] = 0$. These two categories are related by inclusion and its right adjoint Γ_c where $\Gamma_c M$ is the submodule of elements annihilated by some power of c

$$i : \text{tors-}R\text{-mod} \rightleftarrows R\text{-mod} : \Gamma_c \ .$$

We will put model structures on the two categories so that this becomes a Quillen equivalence. On the other hand, there is a second model structure on R-mod that arises by comparison with topology (i.e., free rational $\mathbb{T}$-spectra when $k = \mathbb{Q}$); the two structures on R-mod are Quillen equivalent.

We begin by describing the two relevant model structures on R-mod: a projective model and an injective model. Both of them have the same weak equivalences, and these are detected by the basic 0-cell

$$\underline{k} = \text{Cone}(c : \Sigma^{-2} R \longrightarrow R)$$

which is an exterior algebra over R on a class in degree -1 whose differential is multiplication by c. The basic 0-cell $\underline{k}$ has homology the residue field k, but (unlike $\text{Hom}_R(k, \cdot)$) the functor $\text{Hom}_R(\underline{k}, \cdot)$ preserves homology isomorphisms since $\underline{k}$ is free as an R-module. The basic 0-cell $\underline{k}$ embeds in the contractible object $C\underline{k}$, which is the mapping cone of the identity map of $\underline{k}$.

The projective model structure, k-cell-R-mod$_p$, is cofibrantly generated, taking the set of generating cofibrations I to consist of suspensions of $\underline{k} \longrightarrow C\underline{k}$, and the set of generating acyclic cofibrations J to consist of suspensions of $0 \longrightarrow C\underline{k}$. The weak equivalences are maps $X \longrightarrow Y$ with the property that $\text{Hom}_R(\underline{k}, X) \longrightarrow \text{Hom}_R(\underline{k}, Y)$ is a homology isomorphism. The fibrations are surjective maps, and the cofibrations are injective maps which are retracts of relative I-cellular objects. We have been explicit, but this model structure is an instance of a general construction: it is the $\underline{k}$-cellularization of the usual projective model on R-modules as in Proposition A.2.

For the injective model structure, k-cell-R-mod$_i$, the weak equivalences are again maps $X \longrightarrow Y$ with the property that $\text{Hom}_R(\underline{k}, X) \longrightarrow \text{Hom}_R(\underline{k}, Y)$ is a homology isomorphism. We will not give explicit descriptions of either i-fibrations or i-cofibrations, but the i-cofibrations are in particular monomorphisms and the i-fibrations are in particular surjections with kernels which are injective R-modules. This model structure is the $\underline{k}$-cellularization of the usual injective model on R-modules.

The injective and projective model structures are Quillen equivalent using the identity functors. Using, Proposition A.5 this follows by cellularizing the usual equivalence between projective and injective models on the category of all DG R-modules.

For tors-R-mod we must do something slightly different, since $\mathrm{Hom}_R(k, \cdot)$ does not preserve homology isomorphisms, and $\underline{k}$ is not in tors-R-mod. Instead, weak equivalences are homology isomorphisms, cofibrations are monomorphisms, and the fibrant objects are those which are c-divisible. Coproducts preserve torsion modules, so coproducts in the category tors-R-mod are familiar. However, the product in tors-R-mod of a set M_i of torsion modules is $\Gamma_c \prod_i M_i$. This model structure is given in [8, Appendix B], but it also follows from the more general discussion in Sect. 8.4.

Now the (i, Γ_c) adjoint pair gives a Quillen equivalence between the injective model structure on R-mod and the model structure on tors-R-mod. The point is that i-fibrant R-modules are already torsion modules.

Summary 8.1 We have elementary Quillen equivalences

$$\text{tors-}R\text{-mod} \simeq_Q k\text{-cell-}R\text{-mod}_i \simeq_Q k\text{-cell-}R\text{-mod}_p.$$

8.2 The Koszul model of the residue field

We now return to the category of R-modules where R is a commutative DGA over k with $H^*(R) = H^*(BG)$. To get good control over the model structure, it is useful to choose a good model $\underline{k}$ of the residue field k so that $\mathrm{Hom}_R(\underline{k}, \cdot)$ preserves homology isomorphisms.

We use a standard construction from commutative algebra. If B is a graded commutative ring with elements $x_1, \ldots, x_r$ we may form the Koszul complex

$$K(x_1, \ldots, x_n) = \left(\Sigma^{|x_1|} B \stackrel{x_1}{\longrightarrow} B\right) \otimes_B \cdots \otimes_B \left(\Sigma^{|x_n|} B \stackrel{x_r}{\longrightarrow} B\right),$$

which is finitely generated and free as a B-module. If R is a commutative DGA this needs to be adapted slightly; to build the one-element Koszul object we suppose given a cycle $x \in R$ and then define

$$K(x) := \mathrm{Cone}(\Sigma^{|x|} R \stackrel{x}{\longrightarrow} R).$$

Up to equivalence this depends only on the cohomology class of x. In our case R has polynomial cohomology, and we pick cycle representatives x_i for the polynomial generators, and form

$$K(x_1, \ldots, x_n) = K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_r).$$

We note that this may also be formed from R in r steps by taking fibres. Indeed, if we write $K_i = K(x_1, \ldots, x_i)$, then we have $K_0 = R$ and there is a cofibre sequence

$$\Sigma^{|x_i|} K_{i-1} \stackrel{x_i}{\longrightarrow} K_{i-1} \longrightarrow K_i$$

for $i = 1, 2, \ldots r$.

Definition 8.2 The Koszul model of k is the DG-R-module $\underline{k} = K(x_1, x_2, \ldots, x_r)$ for a chosen set of cocycle representatives of the polynomial generators.

The good behaviour of k-cellularization is based on compactness.

Lemma 8.3 *The basic 0-cell $\underline{k}$ is compact in the sense that*

$$\bigoplus_i \mathrm{Hom}_R(\underline{k}, M_i) \longrightarrow \mathrm{Hom}_R\left(\underline{k}, \bigoplus_i M_i\right)$$

is an isomorphism for any R-modules M_i.

Proof As an R-module $\underline{k}$ is free on 2^r generators. The value of a map $f : \underline{k} \longrightarrow M$ is determined by its values on these generators. □

The Koszul model is designed to preserve weak equivalences, and we need to know that a weak equivalence of torsion modules is a homology isomorphism.

Lemma 8.4 *The Koszul model $\underline{k}$ of k preserves homology isomorphisms in the sense that* $\mathrm{Hom}_R(\underline{k}, \cdot)$ *takes homology isomorphisms to homology isomorphisms.*

Proof The object $\underline{k}$ is formed from R in r steps by taking mapping fibres. The object $\mathrm{Hom}_R(\underline{k}, M)$ is therefore formed from $M = \mathrm{Hom}_R(R, M)$ by finitely many operations of taking mapping cofibres. If M is acyclic, so too is $\mathrm{Hom}_R(\underline{k}, M)$. □

Lemma 8.5 *If X and Y have torsion homology then a map $f : X \longrightarrow Y$ is a homology isomorphism if and only if* $\mathrm{Hom}_R(\underline{k}, X) \longrightarrow \mathrm{Hom}_R(\underline{k}, Y)$ *is a homology isomorphism.*

Proof Considering the mapping cone M of f, it suffices to show that if M has torsion homology then $H_*(M) = 0$ if and only if $H_*(\mathrm{Hom}_R(\underline{k}, M)) = 0$. It follows from 8.4 that if M is acyclic then so is $\mathrm{Hom}_R(\underline{k}, M)$.

Conversely, suppose that $\mathrm{Hom}_R(\underline{k}, M)$ is acyclic and $H_*(M)$ is torsion. Let $K_i = K(x_1, \ldots, x_i)$, and argue by downward induction on i that $\mathrm{Hom}_R(K_i, M)$ is acyclic. The hypothesis states that this is true for $i = r$ and the conclusion is the statement that it is true for $i = 0$.

Suppose then that $\mathrm{Hom}_R(K_i, M)$ is acyclic. Since K_i is the fibre of $x_i : K_{i-1} \longrightarrow K_{i-1}$, we conclude that x_i is an isomorphism of $H_*(\mathrm{Hom}_R(K_{i-1}, M))$. However, since $H_*(M)$ is torsion, so is $H_*(\mathrm{Hom}_R(K_{i-1}, M))$, and hence in particular it is x_i-power torsion. An $H^*(BG)$-module H for which $x_i : H \longrightarrow H$ is an isomorphism and $H[1/x_i] = 0$ is zero. Hence $\mathrm{Hom}_R(K_{i-1}, M)$ is acyclic as required. □

8.3 Models of cellular R-modules

We now formally introduce the algebraic model structures we use. If R is a DGA, we may form the projective model structure R-mod$_p$ (see [16, Section 7] or [22]). The weak equivalences are the homology isomorphisms. It is cofibrantly generated by using algebraic disk-sphere pairs. More precisely, we take $S^{n-1} = S^{n-1}(R)$ to be the $(n-1)$st suspension of R, and $D^n(R)$ to be the mapping cone of its identity map. The set I of generating cofibrations consists of the maps $S^{n-1} \longrightarrow D^n$ for $n \in \mathbb{Z}$ and the set J of generating acyclic cofibrations consists of the maps $0 \longrightarrow D^n$ for $n \in \mathbb{Z}$. The fibrations in this model are the surjective maps, and the cofibrations are retracts of relative cell complexes.

The proof may be obtained by adapting [13, Start of 2.3]. More precisely, D^n is still right adjoint to taking the degree n part and S^{n-1} is still right adjoint to taking degree $n-1$ cycles, so that the proof that it is a model structure (i.e., until the end of the proof of [13, 2.3.5]) is unchanged. The analogue of [13, 2.3.6] states that any cofibrant object is projective if the differential is forgotten, and that any DG-module M admitting an inductive filtration

$$0 = F_0M \subseteq F_1M \subseteq F_2M \subseteq \cdots \subseteq \bigcup_n F_nM = M$$

with subquotients consisting of sums of projective modules (i.e., a *cell R-module* in the sense of [17]) is cofibrant. The proof of projectivity is unchanged, and the cofibrancy of cell modules is just as for CW-complexes (i.e., via the homotopy extension and lifting property (HELP) as in [17, 2.2]). Arbitrary cofibrant objects are retracts of cell R-modules. Cofibrations $i : A \longrightarrow B$ are retracts of relative cell modules.

The model category k-cell-R-mod$_p$ is obtained from R-mod$_p$ by cellularizing with respect to k in the sense of [12]. The Koszul model $\underline{k}$ gives a convenient cofibrant model for k, and since all objects are fibrant, the weak equivalences are the maps $p : X \longrightarrow Y$ for which $\mathrm{Hom}_R(\underline{k}, p)$ is a homology isomorphism, the fibrations are the surjective maps as before. The model structure is cofibrantly generated, and the sets I and J can be formed from $\underline{k} = S^0(\underline{k})$ in the same way that I and J were formed from $R = S^0(R)$ above.

Similarly, in the special case that R has zero differential, which is the only case we will need, the injective model R-mod$_i$ is formed as in [13, 2.3.13]. In fact Hovey only deals with rings and chain complexes, but it applies to a graded algebra R with zero differential and DG modules over it. The weak equivalences are the H_*-isomorphisms, and the cofibrations are the injective maps. The fibrations are the surjections with fibrant kernel. We will not identify the fibrant objects explicitly, but they are in particular injective as modules. For the converse, we say that M is *cocellular* if there is a filtration

$$M = \lim_{\leftarrow n} F^n M \longrightarrow \cdots \longrightarrow F^2 M \longrightarrow F^1 M \longrightarrow F^0 M = 0$$

where the kernels $K^n = \ker(F^n M \longrightarrow F^{n-1} M)$ are injective modules with zero differential. A cocellular R-module is fibrant. The two places where it is necessary to adapt the proof are the proof of [13, 2.3.17] where bounded above modules are replaced by cocellular modules (and the proof with elements replaced by applications of the HELP), and in [13, 2.3.22]. The zero differential allows us to treat any R-module as a DG-module with zero differential.

We can form k-cell-R-mod$_i$ by cellularizing with respect to k in the sense of [12]. In this case k is itself cofibrant, so the standard description of the weak equivalences is that they are maps $p : X \longrightarrow Y$ for which $\mathrm{Hom}_R(k, p')$ is a homology isomorphism, where p' is a fibrant approximation of p; this is a little inconvenient, so we use the equivalent condition that $\mathrm{Hom}_R(\underline{k}, p)$ is a homology isomorphism. The fibrations are the same as for R-mod$_i$, and the cofibrations are as required by the lifting property.

It is immediate from the above description of the model structures that we have a Quillen equivalence

$$R\text{-mod}_p \xrightarrow{\simeq} R\text{-mod}_i$$

using the identity maps. By cellularizing this, see Proposition A.5, we obtain a Quillen equivalence

$$k\text{-cell-}R\text{-mod}_p \xrightarrow{\simeq} k\text{-cell-}R\text{-mod}_i .$$

8.4 A model structure on torsion modules

Finally, we consider a polynomial ring R_a on even degree generators $x_1, \ldots, x_r$. A torsion module is one for which every element is annihilated by a power of the augmentation ideal $\mathfrak{m}$ (or equivalently, it is annihilated by some power of each element of $\mathfrak{m}$). There is an adjunction

$$i : \text{tors-}R\text{-mod} \rightleftarrows R\text{-mod} : \Gamma_{\mathfrak{m}} ,$$

where $\Gamma_{\mathfrak{m}} M$ is the submodule of elements annihilated by some power of the augmentation ideal $\mathfrak{m}$.

Lemma 8.6 *The category tors-R_a-mod of DG torsion modules admits an injective model structure with weak equivalences the homology isomorphisms and cofibrations which are the injective maps. The i-fibrations are the surjective maps with i-fibrant kernel.*

Proof One option is to observe that Hovey's proof of [13, 2.3.13] applies to torsion modules. This is exactly as in the previous section except that to construct products and inverse limits one forms them in the category of all R_a-modules and then applies the right adjoint $\Gamma_{\mathfrak{m}}$. Note that this shows in particular that the category of all torsion R_a-modules does have enough injectives.

Another option is to use the method of [8, Appendix B]. For the latter, we need only show that the four steps described there can be completed. In fact we can use precisely the same argument, which reduces most of the verifications to properties of $\mathbb{Q}$-modules using right adjoints to the forgetful functor to vector spaces, together with the fact that $H^*(BG)$ is of finite injective dimension. The right adjoint is given by coinduction, and the fact that enough injectives are obtained in this way follows since any torsion module over a polynomial ring embeds in a sum of copies of the injective hull of the residue field. □

Proposition 8.7 *The $(i, \Gamma_{\mathfrak{m}})$-adjunction induces a Quillen equivalence*

$$\textit{tors-}R_a\textit{-mod}_i \xrightarrow{\simeq} \textit{k-cell-}R_a\textit{-mod}_i.$$

Proof First observe that since the cofibrations are injective maps in both cases and $\Gamma_{\mathfrak{m}}$ is right adjoint to inclusion, $\Gamma_{\mathfrak{m}}$ takes fibrant objects to fibrant objects.

Next we see that the adjunction is a Quillen pair. The fibrations in k-cell-R_a-mod$_i$ are the surjective maps $p : M \longrightarrow N$ with fibrant kernel K. Fibrant objects are in particular injective, and hence we obtain a short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}} K \longrightarrow \Gamma_{\mathfrak{m}} M \longrightarrow \Gamma_{\mathfrak{m}} N \longrightarrow 0.$$

As observed above, since K is fibrant, so is $\Gamma_{\mathfrak{m}} K$. Thus $\Gamma_{\mathfrak{m}}$ takes fibrations to fibrations.

Next, if p is also a weak equivalence then K is weakly contractible. Indeed, from the exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow 0$$

we see that $\mathrm{Hom}_R(\underline{k}, K)$ is acyclic. However,

$$\mathrm{Hom}_R(\underline{k}, K) \simeq \mathrm{Hom}_R(k, K) = \Gamma_{\mathfrak{m}} K$$

and hence $H_*(\Gamma_{\mathfrak{m}} K) = 0$ as required.

Finally, we check that the Quillen pair is a Quillen equivalence. For this suppose M is a torsion module and N is fibrant. A map $p : M \longrightarrow N$ is a weak equivalence if and only if the map

$$\mathrm{Hom}_R(\underline{k}, M) \longrightarrow \mathrm{Hom}_R(\underline{k}, N)$$

is a homology isomorphism. Now N and $\Gamma_{\mathfrak{m}} N$ are injective, so the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(\underline{k}, N) & \stackrel{\simeq}{\longleftarrow} & \mathrm{Hom}_R(k, N) \\ \uparrow & & \uparrow = \\ \mathrm{Hom}_R(\underline{k}, \Gamma_{\mathfrak{m}} N) & \stackrel{\simeq}{\longleftarrow} & \mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}} N) \end{array}$$

allows us to deduce this is equivalent to the map $\mathrm{Hom}_R(\underline{k}, M) \longrightarrow \mathrm{Hom}_R(\underline{k}, \Gamma_{\mathfrak{m}} N)$ being a homology isomorphism, and finally, by Lemma 8.5, this is equivalent to requiring that

$$M \longrightarrow \Gamma_{\mathfrak{m}} N$$

is a homology isomorphism as required. □

8.5 Rigidity of the category of torsion modules

In this subsection we give the algebraic part of the string of equivalences, following the pattern in Sect. 8.1.

The outcome of the topological part of the argument is the category k_t-cell-R_t-mod$_p$. The only thing we know about R_t is that it is a commutative DGA over $\mathbb{Q}$ with $H^*(R_t) = H^*(BG)$, and the only thing we know about the object we have used to cellularize it is that its homology agrees with that of a free cell.

However this data is rigid in the following sense.

Proposition 8.8 *If R is a commutative DGA over k with cohomology R_a polynomial on even degree generators, and if we cellularize with respect to any compact object k' with homology k, there are Quillen equivalences*

$$k'\text{-cell-}R\text{-mod}_p \simeq k\text{-cell-}R_a\text{-mod}_p \simeq k\text{-cell-}R_a\text{-mod}_i \simeq tors\text{-}R_a\text{-mod}.$$

Proof We have described the model structures on all four categories.

For the first equivalence we have a well-known observation.

Lemma 8.9 *A polynomial ring R_a is intrinsically formal amongst commutative DGAs over k in the sense that there is a homology isomorphism $R_a \longrightarrow R$.*

Proof We choose representative cycles for the polynomial generators. Since R is commutative we may use them to define a map $R_a \longrightarrow R$ of DGAs by taking each polynomial generator to the representative cycle. By construction it is a homology isomorphism. □

Extension and restriction of scalars therefore define a Quillen equivalence

$$R\text{-mod}_p \simeq R_a\text{-mod}_p.$$

We next need to cellularize this equivalence. On the left we use an object with homology equal to k, and the corresponding object on the right has the same property. The second ingredient in rigidity is that this characterizes the objects up to equivalence.

Lemma 8.10 *If M is any R-module with $H_*(M) \cong k$ there is an equivalence $M \simeq k$.*

Proof Since the natural map $\underline{k} \longrightarrow k$ is a homology isomorphism, it suffices to construct a homology isomorphism $\underline{k} \longrightarrow M$. Writing $K_i = K(x_1, \ldots, x_i)$ as before, so that $K_0 = R$ and $K_r = \underline{k}$, there is a cofibre sequence

$$\Sigma^{|x_i|} K_{i-1} \longrightarrow K_{i-1} \longrightarrow K_i$$

for $i = 1, 2, \ldots, n$. Furthermore, since $x_1, \ldots, x_r$ is a regular sequence, $H_*(K_i) = H^*(BG)/(x_1, \ldots, x_i)$. We now successively construct maps $f_i : K_i \longrightarrow M$ for $i = 0, 1, \ldots r$ inducing epimorphisms

$$H^*(K_i) = H^*(BG)/(x_1, \ldots, x_i) \longrightarrow H_*(M) = k$$

in homology. Indeed, we choose a non-zero element of homology to give $K_0 = R \longrightarrow M$, and we may extend $K_{i-1} \longrightarrow M$ over K_i, since $x_i f_{i-1}$ is zero in homology and therefore nullhomotopic. □

Accordingly, by A.5, the Quillen equivalence

$$R_t\text{-mod} \simeq R_a\text{-mod}$$

induces a Quillen equivalence

$$k_t\text{-cell-}R_t\text{-mod}_p \simeq k\text{-cell-}R_a\text{-mod}_p.$$

Combining this with the Quillen equivalences of Sects. 8.3 and 8.4 completes the proof. □

9 Change of groups

In this section we consider restriction of equivariance and its adjoints. Suppose then that G is a compact Lie group and that H is a closed subgroup, with inclusion

$$i : H \longrightarrow G.$$

The restriction functor

$$i^* = \text{res}^G_H : G\text{-spectra} \longrightarrow H\text{-spectra}$$

has a left adjoint and a right adjoint. The left adjoint is induction

$$i_* = \text{ind}^G_H(Y) = G_+ \wedge_H Y$$

and the right adjoint is coinduction

$$i_! = \text{coind}^G_H(Y) = F_H(G_+, Y).$$

Furthermore, these are related by an equivalence

$$F_H(G_+, Y) \simeq G_+ \wedge_H (S^{-L(H)} \wedge Y) \qquad [18, \text{VI. } 1.4]$$

where $L(H)$ is the representation of H on the tangent space to G/H at the identity coset. This suspension already signals the greater sophistication of the right adjoint. When we restrict attention to free spectra and H is connected, suspension by $L(H)$ is equivalent to a G-fixed suspension of the same dimension.

Now consider the situation in which we have given algebraic models, where G and H are connected. We will write $r : H^*(BG) \longrightarrow H^*(BH)$ for the map induced by i to avoid conflict with the use of i^* above.

It can be seen from variance alone that there will be some interesting phenomena involved in finding counterparts in the algebraic model. We begin by motivating the answer with an outline discussion.

Note that at the strict level, restriction

$$r^* : H^*(BH)\text{-mod} \longrightarrow H^*(BG)\text{-mod}$$

has left and right adjoints

$$r_*, r_! : H^*(BG)\text{-mod} \longrightarrow H^*(BH)\text{-mod}$$

defined by extension and coextension of scalars

$$r_*(M) = H^*(BH) \otimes_{H^*(BG)} M \quad \text{and} \quad r_!(M) = \mathrm{Hom}_{H^*(BG)}(H^*(BH), M).$$

The functor corresponding to res_H^G for spectra has the variance of r_* and $r_!$, and it turns out to correspond to $r_!$. Thus *induction* $i_* = \mathrm{ind}_H^G$ (which is the leftmost of the adjoints on spectra) corresponds to *restriction* r^* (which is the middle adjoint in the algebraic world). The counterpart in algebra of the right adjoint $i_!$ is a functor we have not yet mentioned, and it is only easily apparent at the derived level. The existence of this counterpart of the right adjoint $i_!$ at the derived level depends critically on the fact that $H^*(BH)$ is a finitely generated $H^*(BG)$-module by Venkov's theorem. Moving to derived categories, since $H^*(BG)$ is a polynomial ring, the finite generation means that $H^*(BH)$ is small. Accordingly, if we write

$$DM = \mathrm{Hom}_{H^*(BG)}(M, H^*(BG))$$

for the derived dual of a module, smallness means that the natural map

$$r_!(N) = \mathrm{Hom}_{H^*(BG)}(H^*(BH), N) \xrightarrow{\simeq} D(H^*(BH)) \otimes_{H^*(BG)} N$$

is an equivalence in the derived category. It is then apparent that the right adjoint of the derived functor $r_!$ is the functor

$$r^!(M) = \mathrm{Hom}_{H^*(BG)}(D(H^*(BH)), M).$$

Theorem 9.1 *If G and H are connected compact Lie groups, and the inclusion $i : H \longrightarrow G$ induces $r : H^*(BG) \longrightarrow H^*(BH)$ then at the level of homotopy categories*

- *induction of spectra corresponds to restriction of scalars along r.*
- *restriction of free G-spectra corresponds to coextension of scalars along r*
- *coinduction of spectra corresponds to the functor $r^!$*

More precisely, for the left adjoint of restriction, we have Quillen adjunctions

$$\mathrm{ind}_H^G : \textit{free-}H\textit{-spectra} \rightleftarrows \textit{free-}G\textit{-spectra} : \mathrm{res}_H^G$$

and

$$r^* : \textit{tors-}H^*(BH)\textit{-mod} \rightleftarrows \textit{tors-}H^*(BG)\textit{-mod} : r_!$$

where

$$r_!(M) = \mathrm{Hom}_{H^*(BG)}(H^*(BH), M).$$

The resulting derived adjunctions correspond under the Quillen equivalence of Theorem 1.1 in the sense that for each step in the zig-zag of Quillen equivalences of the domain and codomain, there are adjunctions which correspond at the derived level.

For the right adjoints, we have Quillen adjunctions

$$\mathrm{res}^G_H : \textit{free-}G\textit{-spectra} \rightleftarrows \textit{free-}H\textit{-spectra} : \mathrm{coind}^G_H$$

and

$$r'_! : \textit{tors-}H^*(BG)\textit{-mod} \rightleftarrows \textit{tors-}H^*(BH)\textit{-mod} : r^! \ ,$$

where

$$r'_!(M) = DH^*(BH) \otimes_{H^*(BG)} M \simeq \mathrm{Hom}_{H^*(BG)}(H^*(BH), M) =: r_!$$

and

$$r^!(N) = \mathrm{Hom}_{H^*(BH)}(DH^*(BH), N).$$

The associated derived adjunctions of these pairs also correspond at the derived level.

Proof We are in the situation of having a number of model categories $\mathbb{C}(G)$ associated to G, and corresponding model categories $\mathbb{C}(H)$ associated to H. Below we show that certain diagrams

$$\begin{array}{ccc} \mathbb{C}(G) & \xrightarrow{\simeq} & \mathbb{D}(G) \\ {\scriptstyle i_{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle i_{\mathbb{D}}} \\ \mathbb{C}(H) & \xrightarrow{\simeq} & \mathbb{D}(H) \end{array}$$

commute, where the horizontals are Quillen equivalences and the verticals are left adjoints. The relevant categories $\mathbb{C}(G)$ and functors $i : \mathbb{C}(H) \longrightarrow \mathbb{C}(G)$ are

(i) $\mathbb{C}_1(G) =$ free-G-spectra, $i(Y) = G_+ \wedge_H Y$
(ii) $\mathbb{C}_2(G) = \mathbb{Q}[G]$-mod, $i(Y) = \mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} Y$
(iii) $\mathbb{C}_3(G) = k$-cell-R'_{top}-mod, $i(N) = (r'_{\mathrm{top}})^*(N)$
(iv) $\mathbb{C}_4(G) = k$-cell-R_{top}-mod, $i(N) = r^*_{\mathrm{top}}(N)$
(v) $\mathbb{C}_5(G) = \mathbb{Q}$-cell-$R_t$-mod, $i(N) = r_t^*(N)$
(vi) $\mathbb{C}_6(G) = \mathbb{Q}$-cell-$H^*(BG)$-mod$_p$, $i(N) = r_a^*(N)$
(vii) $\mathbb{C}_7(G) = \mathbb{Q}$-cell-$H^*(BG)$-mod$_i$, $i(N) = r_a^*(N)$
(viii) $\mathbb{C}_8(G) =$ tors-$H^*(BG)$-mod, $i(N) = r_a^*(N)$

The commutativity is clear for the diagrams involved in the comparisons 1–2, 3–4, 6–7 and 7–8. The algebraicization step 4-5 also follows since the equivalences of [26] are natural. This leaves 2–3 and 5–6.

For the rigidity step 5–6, we need to take care about the order of choices. Indeed, we choose $r_t = i_t^* : R_t(G) \longrightarrow R_t(H)$ to be a fibration of DGAs. Now consider the square

$$\begin{array}{ccc} H^*(BG) & \cdots\cdots & R_t(G) \\ \downarrow & & \downarrow \\ H^*(BH) & \longrightarrow & R_t(H) \end{array}$$

We obtain the lower horizontal by choosing cycle representatives $\tilde{y}_1, \ldots, \tilde{y}_s$ for the polynomial generators of $H^*(BH) = \mathbb{Q}[y_1, \ldots, y_s]$. Now if $H^*(BG) = \mathbb{Q}[x_1, \ldots, x_r]$ and we choose cycle representatives $\tilde{x}'_1, \ldots, \tilde{x}'_r$ for the polynomial generators, we will not obtain a commutative square. However if $i^*(x_s) = X_s(y)$, the classes $i^*(\tilde{x}_s')$ and $X_s(\tilde{y})$ are cohomologous, and therefore the differences $i^*(\tilde{x}_s') - X_s(\tilde{y})$ are coboundaries. Since the map $R_t(G) \longrightarrow R_t(H)$ is a fibration, we may lift this coboundary to a coboundary $d(e_i)$ in $R_t(H)$. Now take $\tilde{x}_s = \tilde{x}_s' - e_s$ and we complete the square to a commutative square of DGAs.

Finally, the most interesting step is the Koszul step 2–3. The map $i : \mathbb{Q}[H] \longrightarrow \mathbb{Q}[G]$ induces the restriction map

$$r'_{\mathrm{top}} = i^*_{\mathrm{top}} : R'_{\mathrm{top}}(G) = F_{\mathbb{Q}[G]}(k,k) \longrightarrow F_{\mathbb{Q}[H]}(k,k) = R'_{\mathrm{top}}(H)$$

where here k is $\mathbb{Q}[EG]$, the G-cofibrant replacement of $\mathbb{Q}$ which is also an H-cofibrant replacement, since we may choose EG as our model for EH. We must show the diagram

$$\begin{array}{ccc} \mathbb{Q}[G]\text{-mod} & \xrightarrow{k\otimes_{\mathbb{Q}[G]}(\cdot)} & k\text{-cell-}R'_{\mathrm{top}}(G)\text{-mod} \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle r^*} \\ \mathbb{Q}[H]\text{-mod} & \xrightarrow{k\otimes_{\mathbb{Q}[H]}(\cdot)} & k\text{-cell-}R'_{\mathrm{top}}(H)\text{-mod} \end{array}$$

commutes. Starting with a $\mathbb{Q}[H]$-module Y we see that this amounts to the isomorphism

$$k \otimes_{\mathbb{Q}[G]} (Q[G] \otimes_{\mathbb{Q}[H]} Y) \cong k \otimes_{\mathbb{Q}[H]} Y.$$

The left adjoints for the categories (i), (ii), (vii) and (viii) are left Quillen functors and hence induce associated derived functors. For the standard model structures in (iii) through (vi) the functors r^* preserve all weak equivalences, so they also induce derived functors. This shows that the derived functors of the left adjoints for the categories in (i) and in (viii) correspond. Thus their derived right adjoints also correspond.

Now consider the second Quillen pairs listed in the statement of the theorem. The adjunction described at the level of spectra (i.e., $\mathbb{C}_1$) is a Quillen pair by [18, V.2.2]. In algebra (i.e., $\mathbb{C}_8$), $r'_!$ preserves homology isomorphisms and injections, hence it is a left Quillen functor. As shown above, the statement of the theorem, at the derived level $r_!$ is naturally isomorphic to $r'_!$. Since the first statement in the theorem shows that at the derived level res^G_H corresponds to $r_!$, it follows that the second derived adjunctions also correspond. □

Appendix A: Cellularization of model categories

Throughout the paper we need to consider models for categories of cellular objects, thought of as constructed from a basic cell using coproducts and cofibre sequences. These models are usually obtained by the process of cellularization (or colocalization) of model categories, with the cellular objects appearing as the cofibrant objects. Because it is is fundamental to our work, we recall some of the basic definitions from [12].

Definition A.1 [12, 3.1.8] Let $\mathbb{M}$ be a model category and A be an object in $\mathbb{M}$. A map $f : X \to Y$ is an *A-cellular equivalence* if the induced map of homotopy function complexes [12, 17.4.2] $f_* : \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$ is a weak equivalence. An object W is *A-cellular* if W is cofibrant in $\mathbb{M}$ and $f_* : \mathrm{map}(W, X) \to \mathrm{map}(W, Y)$ is a weak equivalence for any A-cellular equivalence f.

Proposition A.2 [12, 5.1.1] *Let $\mathbb{M}$ be a right proper model category which is cellular in the sense of* [12, 12.1.1] *and let A be an object in $\mathbb{M}$. The A-cellularized model category A-cell-$\mathcal{M}$ exists and has weak equivalences the A-cellular equivalences, fibrations the* *fibrations in $\mathbb{M}$ and cofibrations the maps with the left lifting property with respect to the trivial fibrations. The cofibrant objects are the A-cellular objects.*

It is useful to have the following further characterization of the cofibrant objects.

Proposition A.3 [12, 5.1.5] *If A is cofibrant in $\mathbb{M}$, then the class of A-cellular objects agrees with the smallest class of cofibrant objects in $\mathbb{M}$ that contains A and is closed under homotopy colimits and weak equivalences.*

Since we are always working with stable model categories here, homotopy classes of maps out of A detects trivial objects. That is, in $\text{Ho}(A\text{-cell-}\mathcal{M})$, an object X is trivial if and only if $[A, X]_* = 0$. In this case, by [23, 2.2.1] we have the following.

Proposition A.4 *If $\mathbb{M}$ is stable and A is compact, then A is a generator of $\text{Ho}(A\text{-cell-}\mathcal{M})$. That is, the only localizing subcategory containing A is $\text{Ho}(A\text{-cell-}\mathcal{M})$ itself.*

We next need to show that appropriate cellularizations of these model categories preserve Quillen equivalences.

Proposition A.5 *Let $\mathbb{M}$ and $\mathbb{N}$ be right proper cellular model categories with $F : \mathbb{M} \to \mathbb{N}$ a Quillen equivalence with right adjoint U.*

(i) *Let A be an object in $\mathbb{M}$, then F and U induce a Quillen equivalence between the A-cellularization of $\mathbb{M}$ and the FQA-cellularization of $\mathbb{N}$ where Q is a cofibrant replacement functor in $\mathbb{M}$.*

$$A\text{-cell-}\mathcal{M} \simeq_Q FQA\text{-cell-}\mathbb{N}$$

(ii) *Let B be an object in $\mathbb{N}$, then F and U induce a Quillen equivalence between the B-cellularization of $\mathbb{N}$ and the URB-cellularization of $\mathbb{M}$ where R is a fibrant replacement functor in $\mathbb{N}$.*

$$URB\text{-cell-}\mathbb{M} \simeq_Q B\text{-cell-}\mathbb{N}$$

In [14, 2.3] Hovey gives criteria for when localizations preserve Quillen equivalences. Since cellularization is dual to localization, this proposition follows from the dual of Hovey's statement.

Proof First note that $FQURB \to B$ is a weak equivalence since F and U form a Quillen equivalence. The criterion in [12, 3.3.18(2)] (see also [14, 2.2]) for showing that F and U induce a Quillen adjoint pair on the cellular model categories follows in (i) automatically and in (ii) since $FQURB$ is weakly equivalent to B. Similarly, in each case the choice of the cellularization objects establishes the criterion for Quillen equivalences in [14, 2.3].

In more detail, dual to the proof of [14, 2.3], we note that for every cofibrant object X in $\mathbb{N}$, the map $X \to URFX$ is a weak equivalence. Since fibrant replacement does not change upon cellularization and cellular cofibrant objects are also cofibrant in $\mathbb{N}$, we apply [13, 1.3.16] to the Quillen adjunctions of the cellular model categories. Thus, in each case we only need to show that U reflects weak equivalences between fibrant objects. In case (i), we are given $f : X \to Y$ a weak equivalence between fibrant objects such that Uf is an A-cellular equivalence in $\mathbb{M}$. Thus $Uf_* : \text{map}(A, UX) \to \text{map}(A, UY)$ is an equivalence. By [12, 17.4.15], it follows that $f_* : \text{map}(FQA, X) \to \text{map}(FQA, Y)$ is also a weak equivalence. Thus f is a FQA-cellular equivalence as required. In case (ii), we are given $f : X \to Y$ a weak equivalence between fibrant objects such that Uf is an URB-cellular equivalence in $\mathbb{M}$. Thus $Uf_* : \text{map}(URB, UX) \to \text{map}(URB, UY)$ is an equivalence. By [12, 17.4.15], it follows that $f_* : \text{map}(FQURB, X) \to \text{map}(FQURB, Y)$ is also a weak equivalence. Since $FQURB \to RB$ is a weak equivalence, we see that f_* is a B-cellular equivalence as required. □

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