Cotorsion pairs and model categories

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The purpose of this paper is to describe a connection between model categories, a structure invented by algebraic topologists that allows one to introduce the ideas of homotopy theory to situations far removed from topological spaces, and cotorsion pairs, an algebraic notion that simultaneously generalizes the notion of projective and injective objects. In brief, a model category structure on an abelian category $\mathcal{A}$ that respects the abelian structure in a simple way is equivalent to two compatible complete cotorsion pairs on $\mathcal{A}$. This connection enables one to interpret results about cotorsion pairs in terms of homotopy theory (for example, the flat cover conjecture [**4**] can be thought of as the search for a suitable cofibrant replacement), and vice versa.

Besides describing this connection, we also indicate some applications. The stable module category of a finite group G over a field k is a basic object of study in modular representation theory; it is a triangulated category because injective and projective $k[G]$-modules coincide. Cotorsion pairs can be used to construct two different model structures on the category of $K[G]$-modules where K is a commutative Gorenstein ring (such as $\mathbb{Z}$, for example). The homotopy category of these two different model structures is the same; it is a triangulated category that can reasonably be called the stable module category of G over K. This opens up the possibility of "integral representation theory" along the lines of modular representation theory.

Another application is due to Jim Gillespie. In algebraic geometry, a common object of study is the derived category of a scheme, obtained from chain complexes of quasi-coherent sheaves by inverting maps that induce isomorphisms on homology. There is a derived tensor product on this derived category if the scheme is nice enough, but the construction used in algebraic geometry seems to the author to be somewhat ad hoc and difficult to work with. The essential difficulty is that there are not enough projective quasi-coherent sheaves in general. Gillespie has proved a general theorem about promoting a cotorsion pair on an abelian category to a model structure

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on chain complexes over that category. When applied to quasi-coherent sheaves, it produces a model structure compatible with the tensor product of chain complexes of sheaves. The existence of the derived tensor product and its expected properties now follow formally from this model structure.

This paper is an expanded version of two talks given by the author at the Summer School on the Interactions between Homotopy Theory and Algebra at the University of Chicago, July 26 to August 6, 2004. For more details, the reader can consult the papers [**23**], [**16**], and [**15**]. The connection between model structures and cotorsion pairs is also discussed by Beligiannis and Reiten in [**2**, Chapter VIII]. The author would like to thank the organizers of the Summer School for inviting him to speak. The author also thanks the referee for pointing out a subtlety with hereditary cotorsion pairs that the author and Gillespie had both missed.

Contents

1. Cotorsion pairs

Cotorsion pairs were invented by Luigi Salce [**29**] in the category of abelian groups, and were rediscovered by Ed Enochs and coauthors in the 1990's. A **cotorsion pair** in an abelian category $\mathcal{A}$ is a pair $(\mathcal{D}, \mathcal{E})$ of classes of objects of $\mathcal{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. That is, we have

(1) $D \in \mathcal{D}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $E \in \mathcal{E}$; and
(2) $E \in \mathcal{E}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $D \in \mathcal{D}$.

Cotorsion pairs have been used to study covers and envelopes [**13**], [**12**], particularly in the proof of the flat cover conjecture [**4**]. They have also been used in tilting theory [**1**] and in the representation theory of Artin algebras [**26**].

The most obvious example of a cotorsion pair is when $\mathcal{D} = \mathcal{A}$, in which case $\mathcal{E}$ is the class of injective objects. Similarly, we could let $\mathcal{E} = \mathcal{A}$, in which case $\mathcal{D}$ is the class of projective objects.

Based on this example, we say that a cotorsion pair $(\mathcal{D}, \mathcal{E})$ **has enough projectives** if for all X in our abelian category $\mathcal{A}$ there is a short exact sequence

$$0 \to E \to D \to X \to 0$$

where $D \in \mathcal{D}$ and $E \in \mathcal{E}$. So $\mathcal{A}$ has enough projectives in the usual sense if and only if the cotorsion pair (projectives, everything) has enough projectives. On the other hand, the cotorsion pair (everything, injectives) always has enough projectives. Dually, we say that $(\mathcal{D}, \mathcal{E})$ **has enough injectives** if for all X in $\mathcal{A}$ there is a short exact sequence

$$0 \to X \to E \to D \to 0$$

with $E \in \mathcal{E}$ and $D \in \mathcal{D}$. If $(\mathcal{D}, \mathcal{E})$ has enough projectives and enough injectives, we say that it is a **complete cotorsion pair**.

Perhaps the most useful cotorsion pair, and the one that gives the subject its name, is the flat cotorsion pair. Here $\mathcal{A}$ is the category of R-modules for some ring R, $\mathcal{D}$ is the category of flat R-modules, and $\mathcal{E}$ is what it has to be, the collection of all modules E such that $\operatorname{Ext}^1_R(D, E) = 0$ for all flat D. Such modules are called cotorsion modules.

It is not at all obvious that this is a cotorsion pair, or what cotorsion modules look like. A brief digression may be warranted to describe this important example.

First of all, a short exact sequence

$$0 \to A \to B \to C \to 0$$

is called **pure** if it remains exact upon applying the functor $M \otimes_R (-)$ for any R-module M. My favorite reference for purity and many other algebraic topics is [**27**]; purity is discussed in Section 4J, where it is proved, among other things, that the pure exact sequences are the colimits of split exact sequences, and that any short exact sequence where the right-hand entry C is flat is automatically pure. Purity is of considerable interest to logicians interested in the model theory of modules [**19**].

A module A is **pure injective** if every pure exact sequence with A as the left-hand entry is in fact split. There are lots of these around; most importantly, if M is any R-module, then $M^+ = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is always pure injective. (Note that if M is a left R-module, then M^+ is a right R-module). And every pure injective module is cotorsion (because any short exact sequence that ends in a flat is automatically pure), so this gives us a source of cotorsion modules. Using these facts, we can prove that (flat, cotorsion) is in fact a cotorsion pair. Indeed, it suffices to show that if $\operatorname{Ext}^1(D, E) = 0$ for all cotorsion E, then D is flat. But this means that $\operatorname{Ext}^1(D, M^+) = 0$ for all M. Using the derived version of the Hom and tensor adjointness, and the fact that $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, we see that $\operatorname{Tor}^1(D, M)^+ = 0$ for all M, which implies that D is flat.

It was an open question for a long time whether the cotorsion pair (flat, cotorsion) was complete. This became known as the **flat cover conjecture**. It was eventually proved when Eklof and Trlifaj [**8**], working from a mathematical logic perspective, reinvented and applied some model category theoretic techniques (though I don't believe they were aware of the connection). Bican, El Bashir, and Enochs then used the Eklof-Trlifaj result to prove the flat cover conjecture [**4**].

2. Relation between cotorsion pairs and model categories

Recall that a **model category** is a category $\mathcal{M}$, which I will assume has all limits and colimits, together with three subcategories (the **model structure**) called the **weak equivalences**, **cofibrations**, and **fibrations** that must satisfy various axioms. Model categories allow one to export the methods of algebraic topology from topological spaces to more general situations. For the author, the guiding principle is that anytime one has a class of maps that are not isomorphisms but that one wishes were isomorphisms, there should be a model category lurking in the background for which those maps are the weak equivalences. Furthermore, making that model structure explicit often gives rise to additional structures that were not readily apparent beforehand. One of the simplest interesting examples is the category of (unbounded) chain complexes of modules over a ring, where the weak equivalences are the homology isomorphisms. A model category has a homotopy category, obtained by formally inverting the weak equivalences; in the case of chain complexes this homotopy category is known as the derived category of the ring and is of central importance in homological algebra and algebraic geometry. When the ring is commutative, one would like a well-behaved derived tensor product on the derived category. Trying to construct this derived tensor product without a model structure can be quite painful, but with a particular model structure on chain complexes, known as the projective model structure, the existence and the expected properties of the derived tensor product follow easily.

For this paper, we will not need more than the introduction to model categories in [Sections 1-3][**17**] in these proceedings, whose notation we will follow. Another good introduction is [**6**], and reference books on model categories include [**18**], [**20**], and [**21**], as well as the original source [**28**].

Suppose we have a cotorsion pair $(\mathcal{D}, \mathcal{E})$. This means that given any short exact sequence

$$0 \to A \xrightarrow{i} B \to D \to 0$$

with $D \in \mathcal{D}$, and for any $E \in \mathcal{E}$, the map $\mathcal{A}(B, E) \to \mathcal{A}(A, E)$ is surjective, because $\operatorname{Ext}^1(D, E) = 0$. In model category language, this says that given the commutative diagram below

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & 0 \end{array}$$

we can find a lift $g\colon B \to E$ making both triangles commute. This looks like the lifting axiom for model categories [**17**, Definition 1.2]. That is, if we imagine i to be an acyclic cofibration (both a cofibration and a weak equivalence), then $E \to 0$ looks like a fibration, so that $\mathcal{E}$ consists of fibrant objects. This suggests that there should be some relation between model categories and cotorsion pairs.

2.1. Abelian model categories. For this relationship between model categories and cotorsion pairs to hold, we need some relation between the model structure on $\mathcal{A}$ and the abelian structure.

Definition 2.1. An **abelian model category** is an complete and cocomplete abelian category $\mathcal{A}$ equipped with a model structure such that

(1) A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.
(2) A map is a fibration if and only if it is an epimorphism with fibrant kernel.

Here we are using the definition of model structure from [**17**, Section 1.2]; in particular, we will not assume the factorizations in the factorization axiom M.5 are functorial, as was done in [**21**]. Most of the standard model structures on abelian categories are abelian model structures. For example, in the projective model structure on chain complexes, the cofibrations are the monomorphisms with cofibrant (=DG-projective) cokernel, the fibrations are the epimorphisms, and the weak equivalences are the homology isomorphisms. A complex is DG-projective if each entry is projective, and if any map from it to an exact complex is chain homotopic to 0.

A trivial example of a model structure that is not abelian is the one where weak equivalences are isomorphisms and all maps are cofibrations and fibrations (this example is not mentioned in [**17**], but one can easily check it satisfies the axioms). A less trivial example is the absolute model structure on chain complexes [**5**, Example 3.4], where the weak equivalences are chain homotopy equivalences, and everything is cofibrant and fibrant. In this model structure, the cofibrations are the degreewise split monomorphisms and the fibrations are the degreewise split epimorphisms. So an epimorphism with fibrant kernel is usually not a fibration. However, it is possible to modify the definition of abelian model category to include this example and many others, using the idea of a proper class of short exact sequences. This is the same thing as an additive subfunctor of the Ext functor, so there is also a modified definition of a cotorsion pair using this subfunctor. In the case of the absolute model structure, our proper class is the class of degreewise split sequences.

Suppose $\mathcal{A}$ is an abelian model category, and $p\colon X \to Y$ is an acyclic fibration with kernel K. Then $K \to 0$ is a pullback of p, so is also an acyclic fibration. We say that K is an **acyclic fibrant object**, and so an acyclic fibration is an epimorphism with acyclic fibrant kernel. In fact, the converse is true as well; in an abelian model category, every epimorphism with acyclic fibrant kernel is an acyclic fibration. The proof of this converse can be found in [**23**, Proposition 4.2], but it requires Proposition 2.2 below. Dually, an acyclic cofibration is easily seen to be a monomorphism with acyclic cofibrant cokernel; the converse is again less obvious but true.

Now does an abelian model structure have something to do with cotorsion pairs? Yes!

PROPOSITION 2.2. *Suppose $\mathcal{A}$ is an abelian model category. Let $\mathcal{C}$ denote the class of cofibrant objects, $\mathcal{F}$ the class of fibrant objects, and $\mathcal{W}$ the class of acyclic objects* (*those that are weakly equivalent to* 0). *Then* $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ *and* $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ *are complete cotorsion pairs.*

Details of the proof of this proposition can be found in [**23**].

SKETCH OF PROOF. We just do the $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ case as the other is similar. There are 5 steps to the argument.

(1) $\operatorname{Ext}^1(C,K)=0$ for cofibrant C and acyclic fibrant K. An element of $\operatorname{Ext}^1(C,K)$ is represented by a short exact sequence
$$0 \to K \xrightarrow{i} X \xrightarrow{p} C \to 0.$$
Since C is cofibrant, i is a cofibration. By lifting in the diagram
$$\begin{array}{ccc} K & = & K \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & 0 \end{array}$$
we get a splitting of our short exact sequence.

(2) If $\operatorname{Ext}^1(A,K)=0$ for all acyclic fibrant K, then A is cofibrant. Prove this by showing that $\mathcal{A}(A,-)$ takes acyclic fibrations to surjections, so $0 \to A$ has the left lifting property with respect to acyclic fibrations (see [**17**, Remark 1.5(2)] for the definition of the left lifting property).

(3) If $\operatorname{Ext}^1(C,X)=0$ for all cofibrant C, then X is acyclic fibrant. Prove this by showing $X \to 0$ has the right lifting property with respect to cofibrations.

(4) The cotorsion pair has enough projectives. Prove this by factoring $0 \to X$ into a cofibration followed by an acyclic fibration.

(5) The cotorsion pair has enough injectives. Prove this by factoring $X \to 0$ into a cofibration followed by an acyclic fibration.

□

2.2. From cotorsion pairs to an abelian model category. We have seen that an abelian model structure gives rise to two compatible complete cotorsion pairs. Can we go the other way? Well, no, not without some more hypotheses. Recalling the model category axioms [**17**, Definition 1.4], we have the lifting and factorization axioms, but we also have the two out of three axiom and the retract axiom. To make these other axioms work we are going to need some hypothesis on $\mathcal{W}$.

Definition 2.3. A nonempty subcategory of an abelian category is called **thick** if it is closed under retracts and whenever two out of three entries in a short exact sequence are in the thick subcategory, so is the third.

Lemma 2.4. *Suppose $\mathcal{A}$ is an abelian model category and $\mathcal{W}$ is the class of acyclic objects. Then $\mathcal{W}$ is thick.*

We leave the proof to the reader; it can also be found in [**23**].

So now we get the desired theorem.

Theorem 2.5. *Suppose $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{W}$ are three classes of objects in a bicomplete abelian category $\mathcal{A}$, such that*

(1) *$\mathcal{W}$ is thick.*
(2) *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs.*

Then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects, and $\mathcal{W}$ is the class of acyclic objects.

The proof of this theorem (which can be found in [**23**]) is interesting, as it does not follow the usual path for proving something is a model category. Usually the main difficulty is proving the lifting and factorization axioms, but in this case the main difficulty is defining the weak equivalences and proving the two-out-of-three axiom, which is usually trivial.

It is clear that we should define f to be a cofibration if f is a monomorphism with cokernel in $\mathcal{C}$, a fibration if f is an epimorphism with kernel in $\mathcal{F}$, an acyclic cofibration if f is a monomorphism with cokernel in $\mathcal{C} \cap \mathcal{W}$, and an acyclic fibration if f is an epimorphism with kernel in $\mathcal{F} \cap \mathcal{W}$. But weak equivalences do not have to be monomorphisms or epimorphisms, so we can't define them in the same way. Instead, we define f to be a weak equivalence if it is the composition of an acyclic cofibration followed by an acyclic fibration.

There are now a great many things to check. Just to give the flavor of the argument, we prove a few results needed for Theorem 2.5.

Lemma 2.6. *Cofibrations, acyclic cofibrations, fibrations, and acyclic fibrations are all closed under compositions.*

Proof. Suppose $i\colon A \to B$ and $j\colon B \to C$ are cofibrations. We have a short exact sequence

$$0 \to \operatorname{cok} i \to \operatorname{cok} ji \to \operatorname{cok} j \to 0.$$

This is a special case of the snake lemma. Because $\mathcal{C}$ is the left half of a cotorsion pair, it is closed under extensions. Thus $\operatorname{cok} ji \in \mathcal{C}$ and so ji is a cofibration. Because $\mathcal{W}$ is thick, if i and j are acyclic cofibrations, so is ji. The fibration case is similar. □

Proposition 2.7. *Every map f can be factored as $f = qj = pi$, where j is a cofibration, q is an acyclic fibration, i is an acyclic cofibration, and p is a fibration.*

Proof. This proceeds in stages. The two cases are similar, so we just do the qj case. We first assume $f\colon A \to B$ is a monomorphism already, with cokernel C. Since $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a complete cotorsion pair, there is a surjection $QC \to C$ where $QC \in \mathcal{C}$, with kernel K in $\mathcal{F} \cap \mathcal{W}$. By taking the pullback, we get a monomorphism $j\colon A \to B'$ with cokernel QC, so j is a cofibration. We also get $q\colon B' \to B$, which is a surjection with kernel K, so an acyclic fibration as required.

Now suppose f is an epimorphism with kernel K. Then we can repeat the same trick, using an embedding $K \to RK$ with $RK \in \mathcal{F} \cap \mathcal{W}$ and cokernel $C \in \mathcal{C}$, and taking the pushout instead of the pullback.

Now, for an arbitrary map f, we write it as the composite

$$A \xrightarrow{i_1} A \oplus B \xrightarrow{f+1_B} B$$

of a monomorphism followed by an epimorphism. Write $f + 1_B = q'j'$, where q' is an acyclic fibration and j' is a cofibration. Then write $j'i_1 = q''j$, where q'' is an acyclic fibration and j is a cofibration. Take $q = q''q'$ to complete the proof. □

Proposition 2.8. *Weak equivalences as defined above are closed under compositions.*

Proof. It suffices to check that a composition of the form ip, where p is an acyclic fibration and i is an acyclic cofibration, can be written $ip = qj$, where q is an acyclic fibration and j is an acyclic cofibration. By the preceding proposition, we can write $ip = qj$, where q is an acyclic fibration and j is a cofibration. This gives us the diagram below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{j} & W & \longrightarrow & \operatorname{cok} j & \longrightarrow & 0 \\ & & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & Y & \xrightarrow[i]{} & Z & \longrightarrow & \operatorname{cok} i & \longrightarrow & 0 \end{array}$$

which leads to the short exact sequence

$$0 \to \ker p \to \ker q \to \ker r \to 0.$$

Since p and q are acyclic fibrations, $\ker p$ and $\ker q$ are in $\mathcal{W}$. Since $\mathcal{W}$ is thick, $\ker r \in \mathcal{W}$. But $\operatorname{cok} i \in \mathcal{W}$ since i is an acyclic cofibration. We conclude that $\operatorname{cok} j \in \mathcal{W}$ since $\mathcal{W}$ is thick, and hence j is an acyclic cofibration as required. □

3. Cofibrant generation

So now we have this correspondence between abelian model categories and compatible pairs of complete cotorsion pairs. We should then ask: given an important property of model categories, how is that property reflected in the compatible complete cotorsion pairs?

For example, when is our abelian model structure cofibrantly generated? Recall that a model structure is **cofibrantly generated** (see [**17**, Section 3.1]) when there is a set I of cofibrations and a set J of acyclic cofibrations such that p is an acyclic fibration if and only if it has the right lifting property with respect to I, and p is a fibration if and only if it has the right lifting property with respect to J. (There is also an additional smallness condition that we omit, because it is automatically satisfied in any standard algebraic category; it is only topologies that make this one hard). The key thing here is that we do not need the entire proper class of cofibrations to detect the acyclic fibrations, but just the **set** I. Cofibrantly generated model categories are much easier to work with than general model categories; for one thing, the cokernels of the generating cofibrations play a somewhat similar role as the spheres do in the algebraic topology of topological spaces.

The translation between abelian model structures and cotorsion pairs basically takes a cofibration to its cokernel, so we define a cotorsion pair $(\mathcal{D}, \mathcal{E})$ to be **cogenerated by a set** when there is a subset $\mathcal{D}'$ of the class $\mathcal{D}$ such that $E \in \mathcal{E}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $D \in \mathcal{D}'$. This definition was actually made by Eklof and Trlifaj [**8**] without knowing anything about model categories.

For example, the (projective, everything) cotorsion pair is cogenerated by 0 in any abelian category, and the (everything, injective) cotorsion pair in the category of left R-modules is cogenerated by the set of all $R/\mathfrak{a}$, where $\mathfrak{a}$ is a left ideal of R. (This is Baer's criterion for injectivity).

Then the following lemma is not difficult.

Lemma 3.1. *If an abelian model category is cofibrantly generated, then the corresponding complete cotorsion pairs* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *and* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *are each cogenerated by a set.*

We would like the converse to be true as well. In fact, we want more than that. Recall that the point of a model category being cofibrantly generated is then Quillen's small object argument [**17**, Theorem 3.5] gives an automatic proof of the factorization axioms that also proves the naturality of these factorizations. So we want to start with two compatible cotorsion pairs, not necessarily complete, but cogenerated by a set, and argue that the cotorsion pairs are automatically complete, and hence we get an abelian model structure. In fact, this seems to be true in practice, but the simplest theorem along these lines requires a strong hypothesis.

Proposition 3.2. *If $\mathcal{A}$ is a Grothendieck category with enough projectives, then every cotorsion pair cogenerated by a set is complete. Furthermore, given two compatible cotorsion pairs each cogenerated by a set, the corresponding abelian model structure is cofibrantly generated.*

The proof of this proposition can be found in [**23**]. I think of the Grothendieck hypothesis as the best hypothesis on an abelian category. It is general enough to include categories that occur frequently in algebraic topology (sheaves and comodules, for example), but strong enough to ensure good properties. An abelian category is Grothendieck when it is cocomplete and has a generator, and when filtered colimits are exact.

The reason for having projectives is so that, given one of your cogenerators C, you have a good choice for a monomorphism whose cokernel is C. Usually, even when you do not have enough projectives, you actually do have such a good choice anyway, but it is more complicated to make this into a theorem.

4. Monoidal structure

One of the most important properties a model structure can have is compatibility with a tensor product. This is particularly important in the algebraic situation. For example, given any Grothendieck category $\mathcal{A}$, there is an injective model structure on unbounded chain complexes over $\mathcal{A}$. The cofibrations are the monomorphisms, the weak equivalences are the homology isomorphisms, and the fibrations are the epimorphisms with DG-injective kernel. (DG-injective means each entry is injective, and every map from an exact complex into it is chain homotopic to 0). The homotopy category of the injective model structure is the derived category of $\mathcal{A}$, and so the injective model structure is the foundation for homological algebra of the Ext sort in any Grothendieck category.

But as a practical matter, one almost always has a tensor product around; the tensor product of modules, or sheaves, or comodules. And injective resolutions are almost never compatible with the tensor product, which means that one cannot use the injective model structure to produce a derived tensor product on the derived category of $\mathcal{A}$.

In general, we have the following definition.

Definition 4.1. A model structure on a symmetric monoidal category $\mathcal{A}$ is called **monoidal** whenever the following conditions hold:

(1) Given cofibrations $i\colon A \to B$ and $j\colon C \to D$, the induced map

$$i \,\square\, j\colon (A \otimes D) \amalg_{A\otimes C} (B \otimes C) \to B \otimes D$$

is a cofibration, which, in addition, is an acyclic cofibration if either i or j is acyclic.

(2) An annoying condition that only arises when the unit of the tensor product is not cofibrant (see [**21**, Definition 4.2.6]).

The main point of a monoidal model category is that it gives the tensor product on $\mathcal{A}$ homotopy-theoretic meaning. Thus the homotopy category of a monoidal model category $\mathcal{A}$ will itself be a symmetric monoidal category, and one can usually also construct model categories (and thus homotopy categories) of monoids in $\mathcal{A}$ and of modules over a given monoid in $\mathcal{A}$.

The definition of a monoidal model category was not really even formulated precisely until the late 1990's, although it is based on Quillen's definition of a simplicial model category dating to the 1960's (see [**17**, Definition 4.11]). Looking back on it, however, one can say that one of the biggest problems in algebraic topology was the failure to find a monoidal model category whose homotopy category is the usual stable homotopy category. This problem was solved in the 1990's by Elmendorf, Kriz, Mandell, and May [**9**] and Smith [**24**].

The projective model structure on chain complexes of R-modules is monoidal, but, as mentioned above, the injective model structure is not.

Here is what a monoidal abelian model structure looks like from the cotorsion pair point of view.

Theorem 4.2. *Let $\mathcal{A}$ be an abelian model category,and suppose $\mathcal{A}$ is closed symmetric monoidal. Suppose the following conditions are satisfied*:

1. *Every element of $\mathcal{C}$ is flat.*
2. *If $X, Y \in \mathcal{C}$, then $X \otimes Y \in \mathcal{C}$.*
3. *If $X, Y \in \mathcal{C}$ and one of them is in $\mathcal{W}$, then $X \otimes Y \in \mathcal{W}$.*
4. *The unit S is in $\mathcal{C}$.*

Then $\mathcal{A}$ is a monoidal model category.

Again, the proof of this theorem can be found in [**23**]. Here "flat" means what it usually does. That is, X is flat if the functor $X \otimes (-)$, which is right exact since it is a left adjoint, is actually exact.

5. Standard examples

Having done the work of relating abelian model categories to pairs of complete cotorsion pairs, we now consider the standard examples of model structures on abelian categories.

Perhaps the simplest example of a model category is the category of R-modules when R is a quasi-Frobenius ring. This means that projective and injective modules coincide. The standard example is the group ring $R = k[G]$ of a finite group G over a field k. In this case, we can take $\mathcal{C} = \mathcal{F}$ to be the entire category of R-modules, and take $\mathcal{W}$ to be the class of projective (=injective) modules, which is thick in this unusual case. The two complete cotorsion pairs are then (everything, projective=injective) and (projective=injective, everything). The homotopy category of this model category is called the **stable category of R-modules** and is the main object of study in modular representation theory (as practiced by Benson, Carlson, and Rickard, for example). Two modules M and N are isomorphic in the stable category if there are projective modules P and Q with $M \oplus P \cong N \oplus Q$. The map $M \to M \oplus P$ is a typical acyclic cofibration and the map $N \oplus Q \to N$ is a typical acyclic fibration, but it is a bit difficult to say exactly what a stable equivalence is other than a composite of an acyclic cofibration followed by an acyclic fibration.

Now suppose $\mathcal{A}$ is a Grothendieck category. As mentioned above, there is an injective model structure on $\mathrm{Ch}(\mathcal{A})$, the category of unbounded chain complexes on $\mathcal{A}$. Here $\mathcal{C}$ is everything, $\mathcal{W}$ is the exact complexes, and $\mathcal{F}$ consists of the DG-injective complexes (defined at the beginning of Section 4). A complex is DG-injective and exact if and only if it is actually injective, so the cotorsion pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is (everything, injective). The cotorsion pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is (exact, DG-injective). The homotopy category of the injective model structure is the derived category of $\mathcal{A}$.

The dual thing works for $\mathrm{Ch}(R)$, for R a ring. That is, we take $\mathcal{C}$ to be the DG-projective complexes (defined after Definition 2.1), $\mathcal{F}$ to be everything, and $\mathcal{W}$ to be exact complexes. Again, something that is both DG-projective and exact is actually projective. The homotopy category of the projective model structure is the derived category of R, just like the injective model structure. However, the projective model structure is monoidal when R is commutative and so gives us more structure (a derived tensor product and derived Hom) on the derived category than was apparent from the injective model structure.

6. Gorenstein rings

Here is a new example of an abelian model category from [**23**]. The idea here is that we would like to do modular representation theory over the integers instead of over fields. So we want to study $\mathbb{Z}[G]$, when G is a finite group. This is no longer a quasi-Frobenius ring; projectives and injectives do not coincide. However, it does have some exceptionally nice properties: it is left and right Noetherian, and $\mathbb{Z}[G]$, while not self-injective, does have finite injective dimension as either a left or right module over itself. (This was first noticed by Eilenberg and Nakayama [**7**]). Such a ring is called a **Gorenstein ring**, or an **Iwanaga-Gorenstein ring**. It is a reasonable generalization of the usual notion of a commutative Gorenstein ring.

The salient fact about Gorenstein rings is that in a Gorenstein ring, the modules of finite projective dimension and the modules of finite injective dimension coincide (and the maximum injective or projective dimension is the injective dimension of R). This is due to Iwanaga [**25**]. It is easy to prove from this that these modules form a thick subcategory. We then define $\mathcal{W}$ to be this class of modules with finite projective dimension, in analogy to the quasi-Frobenius case.

But now the analogy breaks down a little, as we cannot expect to get a model structure in which every module is both cofibrant and fibrant. If we want every module to be fibrant, then we take $\mathcal{C}$ to be the class of

Gorenstein projective modules; these are, of course, modules P for which $\mathrm{Ext}^1(P,W)=0$ for all W of finite projective dimension. We should point out that Gorenstein projective modules are still interesting over more general rings, but then the definition is necessarily more complex [**12**, Definition 10.2.1]. Let d be the injective dimension of R. Then a typical Gorenstein projective is a dth syzygy of an arbitrary module. That is, if we take a module M and take a partial projective resolution

$$0 \to K \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i are projective, then K is Gorenstein projective. These modules have been studied before; when they are finitely generated, they are called **maximal Cohen-Macaulay modules**.

There is a dual notion of a **Gorenstein injective** module. Here I is Gorenstein injective if and only if $\mathrm{Ext}^1(W,I)=0$ for all W of finite projective dimension. Again, this is only an appropriate definition when the ring itself is Gorenstein; for the general case see [**12**, Definition 10.1.1]. A typical Gorenstein injective module is a dth cosyzygy of an arbitrary module.

We then get, after some work of course, two model structures on the category of R-modules when R is Gorenstein. Both model categories have the same class of acyclic objects $\mathcal{W}$, the modules of finite projective dimension. In the projective model structure, everything is fibrant, and M is cofibrant if and only if it is Gorenstein projective. In the injective model structure, everything is cofibrant, and M is fibrant if and only if it is Gorenstein injective.

The resulting homotopy category (which is the same for both model structures) has every right to be called the **stable category of R-modules**. It is a triangulated category, and when $R=K[G]$ and K is a principal ideal domain, it has a good closed symmetric monoidal structure (given by tensoring over K). It is the natural home for representation theory of G over K.

As far as I know, not very much is known about this stable module category. There are many results about the stable module category of $k[G]$ when k is a field, such as a classification of the thick subcategories of small objects (=finitely generated modules) when G is a p-group [**3**]. It would be good to know how much different the classification over $K[G]$ is.

7. Gillespie's work

The results in this section are due to my student, Jim Gillespie, and come from [**16**], [**15**], and personal communications.

7.1. The general approach. Gillespie looks at the general question of the relationship between a cotorsion pair on a Grothendieck category $\mathcal{A}$ and the homological algebra of $\mathcal{A}$. That is, given a single cotorsion pair $(\mathcal{D},\mathcal{E})$, can we induce a model structure on $\mathrm{Ch}(\mathcal{A})$ from this cotorsion pair on $\mathcal{A}$? We know two cases of this already: the (projective, everything) cotorsion pair on $\mathcal{A}$ corresponds to the projective model structure on $\mathrm{Ch}(\mathcal{A})$, when it exists, and the (everything, injective) cotorsion pair on $\mathcal{A}$ corresponds to the injective model structure on $\mathrm{Ch}(\mathcal{A})$.

Recall how this works for the (projective, everything) model structure. The two cotorsion pairs on $\mathrm{Ch}(\mathcal{A})$ in this case are (projective, everything) and (DG-projective, exact). There is of course a categorical definition of projective in $\mathrm{Ch}(\mathcal{A})$, but that will be of no help for a more general cotorsion pair. Instead, note that a complex X is projective if and only if X is exact and Z_nX is projective for all n. This suggests the following definition.

Definition 7.1. Suppose $\mathcal{D}$ is a class of objects in a bicomplete abelian category $\mathcal{A}$. Define $\widetilde{\mathcal{D}}$ to be the class of objects X in $\mathrm{Ch}(\mathcal{A})$ such that X is exact and $Z_nX \in \mathcal{D}$ for all n.

So if $\mathcal{D}$ is projectives, we recover the notion of a projective complex. If $\mathcal{D}$ is everything, we recover the notion of an exact complex.

We still have to recover the notion of DG-projective. Recall that X is DG-projective if each X_n is projective and any map from X to an exact complex is chain homotopic to 0. This suggests the following definition.

Definition 7.2. Suppoe $(\mathcal{D}, \mathcal{E})$ is a cotorsion pair in a bicomplete abelian category $\mathcal{A}$. Define dg-$\widetilde{\mathcal{D}}$ to be the class of all X in $\mathrm{Ch}(\mathcal{A})$ such that $X_n \in \mathcal{D}$ for all n and every map from X to a complex in $\widetilde{\mathcal{E}}$ is chain homotopic to 0. Similarly, define dg-$\widetilde{\mathcal{E}}$ to be the class of all $X \in \mathrm{Ch}(\mathcal{A})$ such that $X_n \in \mathcal{E}$ for all n and every map from a complex in $\widetilde{\mathcal{D}}$ to X is chain homotopic to 0.

So if $(\mathcal{D}, \mathcal{E})$ is (projectives, everything), then dg-$\widetilde{\mathcal{D}}$ is the class of DG-projectives and dg-$\widetilde{\mathcal{E}}$ is everything. Similarly, if $(\mathcal{D}, \mathcal{E})$ is (everything, injectives), then $\widetilde{\mathcal{D}}$ is the class of exact complexes, $\widetilde{\mathcal{E}}$ is the class of injective complexes, dg-$\widetilde{\mathcal{D}}$ is everything, and dg-$\widetilde{\mathcal{E}}$ is the class of DG-injective complexes.

Now, the goal of Gillespie's work is to prove a metatheorem of the following sort:

Theorem 7.3. *If $(\mathcal{D}, \mathcal{E})$ is a nice enough cotorsion pair on a Grothendieck abelian category $\mathcal{A}$, then there is an induced abelian model structure on $Ch(\mathcal{A})$, where $\mathcal{C} = \text{dg-}\widetilde{\mathcal{D}}$, $\mathcal{F} = \text{dg-}\widetilde{\mathcal{E}}$, and $\mathcal{W}$ is the class of exact complexes.*

Of course, he also wants to give nontrivial examples of this theorem.

Note that, because $\mathcal{W}$ is the category of exact complexes, the homotopy category of any the model structures produced by Theorem 7.3 is the usual derived category of $\mathcal{A}$. So this theorem is not producing new homotopy categories; instead, it is producing new ways to understand the derived category. This is important if one wants the derived category to have some good properties not accessible through the usual injective model structure.

7.2. Making the theorem concrete. We now need to specify precisely what it means for a cotorsion pair $(\mathcal{D}, \mathcal{E})$ to be "nice enough" in Theorem 7.3.

The first thing to verify is that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are indeed cotorsion pairs. This is simple enough that we can do it here, for $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$.

We first show that $\operatorname{Ext}^1(Y, X) = 0$ for $Y \in \widetilde{\mathcal{D}}$ and $X \in \text{dg-}\widetilde{\mathcal{E}}$. So suppose we have a short exact sequence of complexes

$$0 \to X \to W \to Y \to 0$$

with $X \in \text{dg-}\widetilde{\mathcal{E}}$ and $Y \in \widetilde{\mathcal{D}}$. Then each X_n is in $\mathcal{E}$ and each Y_n is in $\mathcal{D}$ (because $Z_nY \in \mathcal{D}$ for all n and Y is exact, so Y_n is an extension of Z_nY and $Z_{n-1}Y$). Therefore, our short exact sequence of complexes is dimensionwise split, so $W_n \cong X_n \oplus Y_n$. In terms of this decomposition, the differential on W is $d = (d_X, \tau + d_Y)$, where $\tau \colon Y_n \to X_{n-1}$. Because $d^2 = 0$, we see that $\tau \colon Y \to \Sigma X$ is a chain map. By hypothesis, this chain map is chain homotopic to 0. The chain homotopy can then be used to define a splitting of our sequence by a chain map.

Now suppose $\operatorname{Ext}^1(Y, X) = 0$ for all $X \in \text{dg-}\widetilde{\mathcal{E}}$. We want to show that $Y \in \widetilde{\mathcal{D}}$. The first thing to point out is that

$$\operatorname{Ext}^1(Y, D^{n+1}A) \cong \operatorname{Ext}^1(Y_n, A).$$

(To see this, just draw what an extension of complexes looks like). It follows easily from this that $Y_n \in \mathcal{D}$ for all n, since $D^nA \in \text{dg-}\widetilde{\mathcal{E}}$ whenever $A \in \mathcal{E}$.

Given this, an element of $\operatorname{Ext}^1(Y, S^{n-1}A)$ is determined by a map

$$Y_n/B_nY \to A$$

(this is the same as a chain map $Y \to S^nA$). However, two maps determine the same extension if they are chain homotopic as chain maps $Y \to S^nA$. Said another way, $\operatorname{Ext}^1(Y, S^{n-1}A)$ is the quotient

$$\operatorname{Hom}(Y_n/B_nY, A)/\operatorname{Hom}(Y_{n-1}, A).$$

If this quotient is to be 0 for all $A \in \mathcal{E}$, we can in particular take A to be an injective object containing Y_n/B_nY to see that Y is exact. But then $\operatorname{Ext}^1(Y, S^{n-1}A)$ is isomorphic to $\operatorname{Ext}^1(Z_{n-1}Y, A)$, from which we see that $Z_{n-1}Y \in \mathcal{D}$ for all n. Thus $Y \in \widetilde{\mathcal{D}}$ as required.

A similar, but simpler, argument shows that if $\operatorname{Ext}^1(Y, X) = 0$ for all $Y \in \widetilde{\mathcal{D}}$, then $X \in \text{dg-}\widetilde{\mathcal{E}}$. For this, one uses the isomorphism

$$\operatorname{Ext}^1(D^nA, X) \cong \operatorname{Ext}^1(A, X_n)$$

to see that $X_n \in \mathcal{E}$ for all n. It then follows that any element in $\operatorname{Ext}^1(Y, X)$ is dimensionwise split for $Y \in \widetilde{\mathcal{D}}$, so $\operatorname{Ext}^1(Y, X)$ is isomorphic to chain homotopy classes of chain maps from Y to ΣX. Since $\operatorname{Ext}^1(Y, X) = 0$, we see that $X \in \text{dg-}\widetilde{\mathcal{E}}$.

Now, if we worked with $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ instead, we would have run into a problem. In the above argument, there was a point where we embedded

Y_n/B_nY into an element of $\mathcal{E}$, which we can do by taking an injective. The dual will cause us trouble because we do not want to assume there are enough projectives in $\mathcal{A}$. So instead we assume there are enough $\mathcal{D}$-objects in $\mathcal{A}$, in the sense that everything in $\mathcal{A}$ is a quotient of something in $\mathcal{D}$. This would be automatic if $(\mathcal{D},\mathcal{E})$ were a complete cotorsion pair.

So we get the following proposition of Gillespie.

PROPOSITION 7.4. *If $(\mathcal{D},\mathcal{E})$ is a cotorsion pair on a Grothendieck category $\mathcal{A}$ that has enough $\mathcal{D}$-objects, then $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are cotorsion pairs on Ch$(\mathcal{A})$.*

We now want to know whether these cotorsion pairs are compatible with the class $\mathcal{W}$ of exact complexes. That is, we want to know that

$$\text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W} = \widetilde{\mathcal{D}} \text{ and } \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W} = \widetilde{\mathcal{E}}.$$

It is fairly straightforward to show the inclusions

$$\widetilde{\mathcal{D}} \subseteq \text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W} \text{ and } \widetilde{\mathcal{E}} \subseteq \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W}.$$

One shows that any map from something in $\widetilde{\mathcal{D}}$ to something in $\widetilde{\mathcal{E}}$ is chain homotopic to 0. The idea for the converse is as follows. Given $X \in \text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W}$, we want to show $\operatorname{Ext}^1(Z_nX, A) = 0$ for all $A \in \mathcal{E}$. Since we have the short exact sequence

$$0 \to Z_{n+1}X \to X_{n+1} \to Z_nX \to 0$$

and $X_{n+1} \in \mathcal{D}$, it suffices to show that any map $Z_{n+1}X \to A$ extends to a map $X_{n+1} \to A$. Take an augmented injective resolution I_* of A (so $I_0 = A$ and I_{-1} is an injective object containing A). With any justice, this should be a complex in $\widetilde{\mathcal{E}}$, since A was in $\mathcal{E}$ to start with. Then a map $Z_{n+1}X \to A$ induces a map of complexes $\Sigma^{-n-2}X \to I_*$ using injectivity. This chain map is chain homotopic to 0, and the chain homotopy gives us an extension $X_{n+1} \to A$.

This argument depended on I_* actually being in $\widetilde{\mathcal{E}}$. This is **NOT** automatic, however. Consider the following three conditions on a cotorsion pair $(\mathcal{D},\mathcal{E})$.

(1) $\operatorname{Ext}^i(D, E) = 0$ for all $D \in \mathcal{D}$, $E \in \mathcal{E}$, and $i > 0$.
(2) $\mathcal{D}$ is closed under kernels of epimorphisms.
(3) $\mathcal{E}$ is closed under cokernels of monomorphisms.

One can check easily that the first condition above implies the second and third; when our cotorsion pair satisfies this first condition, we call it a **hereditary** cotorsion pair. The second condition is equivalent to the first when our category has enough projectives, and the third condition is equivalent to the first when our category has enough injectives.

Most cotorsion pairs that arise naturally are hereditary, though it can sometimes be hard to prove that a cotorsion pair is hereditary if there are not enough projectives in the category.

Then we have the following proposition, again due to Gillespie.

Proposition 7.5. *Suppose* $(\mathcal{D},\mathcal{E})$ *is a hereditary cotorsion pair in a Grothendieck category* $\mathcal{A}$ *with enough* $\mathcal{D}$*-objects. Then* $\text{dg-}\widetilde{\mathcal{D}}\cap\mathcal{W}=\widetilde{\mathcal{D}}$*. If, in addition,* $\mathcal{A}$ *has enough projectives, then* $\text{dg-}\widetilde{\mathcal{E}}\cap\mathcal{W}=\widetilde{\mathcal{E}}$*.*

As a practical matter, though, most of the categories we are interested in do not have enough projectives. Gillespie and Ed Enochs get around this with a subtle transfinite induction argument that gives the following proposition.

Proposition 7.6. *Suppose* $(\mathcal{D},\mathcal{E})$ *is a cotorsion pair that is cogenerated by a set on a Grothendieck category* $\mathcal{A}$ *with enough* $\mathcal{D}$*-objects. Then* $(\text{dg-}\widetilde{\mathcal{D}},\widetilde{\mathcal{E}})$ *has enough injectives.*

One can look on this proposition as a variant of the small object argument, but it is much more complicated to prove. Also, it does not seem to work for $(\widetilde{\mathcal{D}},\text{dg-}\widetilde{\mathcal{E}})$.

From this, then, we get the following proposition of Gillespie.

Proposition 7.7. *Suppose* $(\mathcal{D},\mathcal{E})$ *is a hereditary torsion theory cogenerated by a set on a Grothendieck category* $\mathcal{A}$ *with enough* $\mathcal{D}$*-objects. Then*

$$\text{dg-}\widetilde{\mathcal{D}}\cap\mathcal{W}=\widetilde{\mathcal{D}},\text{dg-}\widetilde{\mathcal{E}}\cap\mathcal{W}=\widetilde{\mathcal{E}},$$

and $(\text{dg-}\widetilde{\mathcal{D}},\widetilde{\mathcal{E}})$ *is complete.*

The proof is not hard. Suppose $X\in\text{dg-}\widetilde{\mathcal{E}}\cap\mathcal{W}$. We have a short exact sequence

$$0\to X\to W\to Y\to 0$$

with $W\in\widetilde{\mathcal{E}}$ and $Y\in\text{dg-}\widetilde{\mathcal{D}}$. But then X and W are exact, so Y is too. Thus $Y\in\widetilde{\mathcal{D}}$. This means the sequence splits, so X is a summand in W. But then $X\in\widetilde{\mathcal{E}}$.

Given that $(\text{dg-}\widetilde{\mathcal{D}},\widetilde{\mathcal{E}})$ has enough injectives, we can use a pushout trick to show it has enough projectives as well, using the fact that there are enough $\mathcal{D}$-objects. That is, you first show that $\text{Ch}(\mathcal{A})$ has enough $\widetilde{\mathcal{D}}$-objects. Then, given X, you take a surjection $A\to X$ with kernel K, where $A\in\widetilde{\mathcal{D}}$. Then you embed K in an element of $\widetilde{\mathcal{E}}$ with cokernel in $\text{dg-}\widetilde{\mathcal{D}}$, and you take the pushout.

So, to complete Gillespie's program, we must ensure that $(\widetilde{\mathcal{D}},\text{dg-}\widetilde{\mathcal{E}})$ is complete. In fact, the pushout trick above shows that we only need to be sure it has enough injectives. This appears to be the heart of the matter.

One always wants to use some version of the small object argument of Quillen. But it just seems to be harder than it is for model categories, and being cogenerated by a set does not seem to be enough. So Gillespie, following Enochs and Lopéz-Ramos [**14**], strengthens the definition a bit.

Definition 7.8. A class $\mathcal{D}$ is a **Kaplansky class** if there is some cardinal κ such that, for every κ-generated subobject T of an object $D\in\mathcal{D}$, there is a κ-presentable object $S\in\mathcal{D}$ such that $T\subseteq S\subseteq D$ and $D/S\in\mathcal{D}$.

In an arbitrary category, an object A is κ-generated if $\operatorname{Hom}(A,-)$ commutes with λ-fold coproducts for all κ-filtered ordinals λ (any regular cardinal larger than κ is κ-filtered). On the other hand, A is κ-presentable if $\operatorname{Hom}(A,-)$ commutes with all κ-filtered colimits. The easiest case is when $\kappa=\omega$, when we do recover the usual definition of finitely generated and finitely presentable, only without reference to a specific generator of the category.

This is a strange definition at first. It is motivated by flat modules, where it asserts that, given any small subset of a flat module, there is a flat submodule that contains it and sits inside the big module purely. This was the key idea in the proof of the flat cover conjecture by Bican, El Bashir, and Enochs [**4**].

We can now state the precise version of Gillespie's main theorem.

THEOREM 7.9. *Suppose $(\mathcal{D},\mathcal{E})$ is a hereditary cotorsion pair cogenerated by a set such that $\mathcal{D}$ is a Kaplansky class on a Grothendieck category $\mathcal{A}$ with enough $\mathcal{D}$-objects. Then there is an induced abelian model structure on Ch($\mathcal{A}$), where $\mathcal{C}=$ dg-$\widetilde{\mathcal{D}}$, $\mathcal{F}=$ dg-$\widetilde{\mathcal{E}}$, and $\mathcal{W}$ is the class of exact complexes.*

7.3. Sheaves and schemes. The motivation for Gillespie's work was to better understand the derived category of sheaves on a ringed space and the derived category of quasi-coherent sheaves on a scheme. Recall that a ringed space is a topological space S equipped with a sheaf of rings $\mathcal{O}$; that is, a contravariant functor from open sets of S to commutative rings that is locally determined (the sheaf property). A one-point ringed space is of course a commutative ring. The category of $\mathcal{O}$-modules is therefore a generalization of the category of R-modules for a commutative ring R; here an $\mathcal{O}$-module M is a sheaf of abelian groups over S such that $M(U)$ is naturally a module over $\mathcal{O}(U)$. The category of $\mathcal{O}$-modules has a lot in common with the category of R-modules; it is a closed symmetric monoidal Grothendieck category. There is a tensor product defined stalkwise in the obvious way.

Therefore, we would expect $\operatorname{Ch}(\mathcal{O})$ to be a symmetric monoidal model category, so that the derived category of $\mathcal{O}$-modules inherits a tensor product. However, before Gillespie's work, I don't believe this was known. The injective model structure certainly exists on $\operatorname{Ch}(\mathcal{O})$, but it is not compatible with the tensor product, and cannot be used to define a derived tensor product. The projective model structure only exists rarely, because generally there are not enough projective $\mathcal{O}$-modules. There are enough flats, though; in fact, the flat sheaves $\mathcal{O}_U$ generate the category, where the stalks of $\mathcal{O}_U$ agree with the stalks of $\mathcal{O}$ inside U and are 0 outside U. The author used these sheaves to construct a monoidal model structure on $\operatorname{Ch}(\mathcal{O})$ in [**22**], but only under an annoying technical assumption on the ringed space, involving the finiteness of sheaf cohomology.

Gillespie's work allows one to use all the flat sheaves at once, rather than just the ones one can explicitly write down. That is, we start with the (flat, cotorsion) cotorsion pair on $\mathcal{O}$-modules. One needs an argument involving the stalks to see that this cotorsion pair is hereditary. Using the approach to the flat cover conjecture of [**4**], Gillespie shows that that the flat $\mathcal{O}$-modules form a Kaplansky class. (The proof involves purity in an essential way). Hence Theorem 7.9 gives us an abelian model structure on $\mathrm{Ch}(\mathcal{O})$, which Gillespie proves is compatible with the tensor product. Gillespie therefore gets a derived tensor product and a derived Hom functor with all the usual properties on the derived category of $\mathcal{O}$-modules.

In algebraic geometry, however, it is more common to use the category of quasi-coherent sheaves on a scheme. This is because if your ringed space is Spec R, then a quasi-coherent sheaf is equivalent to an R-module, whereas an arbitrary sheaf could be more complicated. The word quasi-coherent can be thought of as meaning locally a quotient of free sheaves. We need a special argument to see that the (flat, cotorsion) pair is hereditary, involving comparision to open affine subschemes, that only works if the scheme is quasi-separated. Now Enochs, Estrada, Garcá Rozas, and Oyonarte [**10**] have proved a result equivalent to the fact that flat quasi-coherent sheaves form a Kaplansky class (see also [**11**, Proposition 3.3]). There is an additional complication though; it is much less obvious that there are enough flat quasi-coherent sheaves (the sheaves $\mathcal{O}_U$ are not quasi-coherent). This is known by algebraic geometers when the scheme is quasi-compact and separated, and Gillespie and I suspect it holds when the scheme is quasi-compact and quasi-separated (this seems to be the hypothesis of choice in algebraic geometry anyway). But in any case, Gillespie's work then leads to an abelian monoidal model structure of chain complexes of quasi-coherent sheaves over a quasi-compact, separated scheme, and hence a derived tensor product and a derived Hom functor on the derived category of the scheme. This derived tensor product is used frequently by algebraic geometers, and this provides a simple reason for its existence and a simple proof that it has all the properties one would expect. (I believe the usual approach is to patch together the derived tensor products of each affine piece of the scheme).

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