1. Prove the following statement: Two matrices have the same reduced echelon form if and only if they are row equivalent. (Hint: This was proved quickly in class. Write down the complete argument using equivalence relation properties and quoting statements from the book.)

\[ \Rightarrow: \text{Suppose two matrices } A \text{ and } B \text{ have the same reduced echelon form, } C. \text{ Then } A \sim C \text{ and } B \sim C, \text{ where } \sim \text{ denotes row equivalence. Row equivalence is symmetric and transitive (Lemma 1.5), so } A \sim B. \]

\[ \Leftarrow: \text{Suppose two matrices } A \text{ and } B \text{ are row equivalent, that is } A \sim B. \text{ Both } A \text{ and } B \text{ have unique reduced echelon forms (Theorem 2.7); call them } A' \text{ and } B' \text{ respectively. Then } A \sim A' \text{ and } B \sim B'. \text{ Again using symmetry and transitivity (Lemma 1.5) we have } A' \sim B'. \text{ Thus } A' \text{ and } B' \text{ are both reduced echelon forms of } A. \text{ Hence by uniqueness, } A' = B'. \]

2. Is the vector \( \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} \) in the set of vectors given by \( \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y \) with \( x \) and \( y \) real numbers? (Hint: This can be rewritten as a linear system of equations. Then use Gauss-Jordan reduction to solve the system.)

We want to solve \( \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y. \)

Combine the constant terms to get \( \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}. \)

In matrix form this is \( \begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 & 7 \\ 4 & 2 & 10 \end{pmatrix} \)

which row-reduces to \( \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \)

giving values \( x = 2 \) and \( y = 1. \) Plugging into the original equation,

\( \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot 2 + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot 1 \)

indeed. The answer is ”Yes”.