PROOF BY INDUCTION

BROOKE SHIPLEY

In Math 300, we assume the following theorem as the basis for our proofs by induction.

Theorem 1. (Weak) Induction Principle *Let S be a set of natural numbers. Assume that a is an element of S and that if $n > a$ and n is in S, then $n+1$ is also in S. Then S is the set of natural numbers greater than or equal to a.*

We can use this principle to show that a given statement is true for all natural numbers. This is called "proof by induction."

Example 2. *Prove* $1+3+5+7+\cdots+(2n-1) = n^2$ *for all* $n \in \mathbb{N}$.

Proof. Let S be the set of natural numbers n such that this statement holds.
Step 1: [Base case] Show that $n=1$ is in S. Here both sides of the equation agree for $n=1$, since $2 \cdot 1 - 1 = 1$ and $1^2 = 1$.
Step 2: [Induction step] Assume that n is in S and show that $n+1$ is in S. We assume that $1+3+5+7+\cdots+(2n-1) = n^2$. We want to show this is true for $n+1$; that is, we want to show that $1+3+5+\cdots+(2(n+1)-1) = (n+1)^2$.

Start with the left hand side of the statement for $n+1$, but include the next to last term as well.

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1) = 1+3+5+\cdots+(2n-1)+(2n+1).$$

Here, we simplified the last term. In the next step we use our assumption that n is in S to replace $1+3+5+\cdots+(2n-1)$ on the right hand side by n^2. This shows that

$$1+3+5+\cdots+(2n-1)+(2n+1) = n^2 + (2n+1).$$

Next we recognize the right hand side as $n^2+2n+1 = (n+1)^2$. So, the statement is true for $n+1$; that is, $n+1$ is in S. Thus, by the Induction Principle, S contains all the natural numbers (greater than or equal to 1). Or, in other words, the statement holds for all $n \in \mathbb{N}$. □

After the first few induction proofs we usually use the Induction Principle more implicitly. Feel free to follow either the example above or the one below in your proofs by induction.

Proof. **Step 1: [Base case]** Show that the statement holds for $n = 1$. Here both sides of the equation agree for $n = 1$, since $2 \cdot 1 - 1 = 1$ and $1^2 = 1$.
Step 2: [Induction step] Assume the statement holds for n; that is, $1+3+5+7+\cdots+(2n-1) = n^2$. We need to show this is true for $n+1$; that is, we want to show that $1+3+5+\cdots+(2(n+1)-1) = (n+1)^2$.

Start with the left hand side of the statement for $n+1$, but explicitly include the next to last term as well.

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1) = 1+3+5+\cdots+(2n-1)+(2n+1).$$

Here, we simplified the last term. In the next step we use our assumption that the statement holds for n to replace $1+3+5+\cdots+(2n-1)$ on the right hand side by n^2. This shows that

$$1+3+5+\cdots+(2n-1)+(2n+1) = n^2+(2n+1).$$

Next we recognize the right hand side here as $n^2+2n+1 = (n+1)^2$. So, the statement is true for $n+1$. Thus, by induction, the statement holds for all $n \in \mathbb{N}$. $\square$

Note that using summation notation the previous example could also be written as:

$$\Sigma_{k=1}^{n}(2k-1) = n^2$$

Exercises due October 24/26.

1. Prove the following for all positive integers n:
$1+5+9+13+\cdots+(4n-3) = \frac{1}{2}n(4n-2)$ or

$$\Sigma_{k=1}^{n}(4k-3) = \frac{1}{2}n(4n-2).$$

2. Prove that n^3+2n is divisible by 3 for all positive integers n.