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Symmetric Spectra and Topological Hochschild Homology

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Abstract. A functor is defined which detects stable equivalences of symmetric spectra. As an application, the definition of topological Hochschild homology on symmetric ring spectra using the Hochschild complex is shown to agree with Bökstedt's original ad hoc definition. In particular, this shows that Bökstedt's definition is correct even for non-connective, non-convergent symmetric ring spectra.

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1. Introduction

The category of symmetric spectra introduced by Jeff Smith is a closed symmetric monoidal category whose associated homotopy category is equivalent to the traditional stable homotopy category, see [11]. Unlike most other categories of spectra, not all stable weak equivalences of symmetric spectra induce π_*-isomorphisms, that is, isomorphisms of the classical stable homotopy groups, $\pi_n X = \mathrm{colim}_k\, \pi_{n+k} X_k$, defined on the underlying prespectra. Hence, another way to identify stable equivalences here is necessary.

To remedy this we consider a detection functor, D, which turns stable equivalences into π_*-isomorphisms. Theorem 3.1.2 shows that $X \to Y$ is a stable equivalence if and only if $DX \to DY$ is a π_*-isomorphism. Thus, the classical stable homotopy groups of DX are invariants of the stable homotopy type of X. In fact, the groups $\pi_* DX$ are the derived stable homotopy groups of X. That is, $\pi_* DX \cong \pi_* LX$, where L is a stable fibrant replacement functor, see 2.1.3.

Basically, the classical stable homotopy groups of X are not homotopy invariants of X because their construction ignores the symmetric group actions on the levels of X. The construction of D modifies the usual telescope or sequential homotopy colimit construction to use the extra symmetric structure. More specifically, $DX_n = \mathrm{hocolim}_{k \in I}\, \Omega^k f \Sigma^n X_k$ where the indexing category is the category of finite sets and injections and f is a fibrant replacement functor, see Definition 3.1.1. As with any homotopy colimit, there is a spectral sequence for calculating $\pi_* DX$, the derived stable homotopy groups of X, see Proposition 2.2.4. This functor D in fact appears in the zeroth level of the simplicial spectrum in Bökstedt's definition of topological Hochschild homology (THH) [1]. This provides the starting point for the comparison of various definitions of THH discussed below.

Although in general stable weak equivalences are not π_*-isomorphisms, in the full subcategory of semistable spectra (see Definition 2.1.6), the stable weak equivalences are exactly the π_*-isomorphisms. In the even more specialized full subcategory of Ω-spectra, the stable weak equivalences are exactly the level equivalences. This detection functor is then analogous to a localization functor; after its first application it localizes a spectrum to give a semistable spectrum and after two applications it produces a spectrum which is level equivalent to an Ω-spectrum, see Theorem 3.1.5. Thus, in model category theoretic terms, D^2 composed with level fibrant replacement is a stable fibrant replacement functor which is more explicit than the one introduced in [11] that relies on the small object argument, see Definition 2.1.3.

As an application of this detection functor we consider the THH of a symmetric ring spectrum. One of the motivations for creating a symmetric monoidal category of spectra was to provide a setting where one could easily mimic the usual constructions of algebra. Now that this is possible, one can verify that the classical ad hoc construction of THH due to Bökstedt, [1], agrees with these simple algebraic definitions. Due to the work of [13, 16], the verification here for symmetric ring spectra also shows that the algebraic definition of THH agrees with Bökstedt's original definition when defined for S-algebras, functors with smash product, Gamma rings, and orthogonal ring spectra.

In Section 4 we give two definitions based on algebra of THH for a symmetric ring spectrum; one uses a derived smash product over the enveloping algebra, 4.1.1, and the other mimics the Hochschild complex, 4.1.2. These two definitions are then shown to agree under certain cofibrancy conditions in Theorem 4.1.10 and to agree with Bökstedt's original definition of THH in Theorem 4.2.8.

Perhaps the most surprising of these results is the agreement of Bökstedt's definition with the others without any of the connectivity or convergence conditions that Bökstedt originally required. Some conditions are indeed necessary to apply Bökstedt's approximation theorem [1, 1.6], though the usual connectivity and convergence conditions can be weakened to include any semistable spectrum, see Corollary 3.1.7. For spectra which are not semistable, the model category structure on symmetric ring spectra is used instead to prove comparison results such as Theorems 3.1.2 and 4.2.8. Also, without any extra conditions, Bökstedt's original definition of THH takes stable equivalences of symmetric ring spectra to π_*-isomorphisms (see Corollary 4.2.9, Remark 2.1.12).

Symmetric spectra have also been used by Jardine in [12] to define a symmetric monoidal model category which gives rise to the Morel–Voevodsky stable homotopy theory [19]. An analysis similar to the one carried out here will be necessary to detect stable equivalences in this setting. The details of the analysis here do not immediately carry over to the stable $\mathbb{A}^1$-local homotopy theory, but the broad outline should in fact generalize.

Outline. In the first section we recall various properties of symmetric spectra from [11] and define symmetric ring spectra. In Section 2.2 we define the homotopy colimit of diagrams of symmetric spectra and state several comparison results for homotopy colimits which are used in Sections 3 and 4. The functor D, which detects stable equivalences, is defined in Section 3. As an application of this detection functor, in Section 4, three different definitions of topological Hochschild homology are defined and compared.

2. Basic Definitions

In the first section, we recall the definitions and properties of symmetric spectra which are essential to this paper. In Section 2.2, we consider the properties of the homotopy colimit needed for Sections 3 and 4.

2.1. SYMMETRIC SPECTRA

First we recall the symmetric spectra, $F_n K$, and some properties of the cofibrantly generated stable model category of symmetric spectra. Then we consider a subcategory of symmetric spectra, the semistable spectra, between which stable equivalences are exactly the π_*-isomorphisms. Finally, we recall the definition of symmetric ring spectra, R-modules, and R-algebras. Throughout this paper 'space' means simplicial set, except in Remark 3.1.4. We denote by $\mathcal{S}_*$ the category of pointed simplicial sets and by Sp^Σ the category of symmetric spectra over simplicial sets.

We first give a slightly different description of the free symmetric spectra $F_n K$ defined in [11, 2.2.5] which play an important role in the model category structures and in the later sections of this paper. Let I be the skeleton of the category of finite sets and injections with objects $\mathbf{n}$. Note that $\hom_I(\mathbf{n}, \mathbf{m}) \cong \Sigma_m / \Sigma_{m-n}$ as Σ_m sets. This isomorphism gives the following proposition. The close connection between the following free spectra and free diagrams over I (see Definition 3.2.6) is the key reason for the use of I in the definition of the detection functor given in the next section.

PROPOSITION 2.1.1. $(F_n K)_m = \Sigma_m^+ \wedge_{\Sigma_{m-n}} S^{m-n} \wedge K \cong \hom_I(\mathbf{n}, \mathbf{m})_+ \wedge S^{m-n} \wedge K$ *where* $S^n = *$ *for* $n < 0$.

F_n is left adjoint to the nth evaluation functor $Ev_n \colon Sp^\Sigma \to \mathcal{S}_*$ where $Ev_n(X) = X_n$. There is a natural isomorphism $F_n K \wedge_S F_m L \to F_{n+m}(K \wedge L)$.

The stable model category of symmetric spectra is cofibrantly generated by [11, 3.4]. In particular, this means that a version of Quillen's small object argument [14, II 3.4] exists. One aspect of the small object argument implies the following proposition, see also [17, 2.1].

PROPOSITION 2.1.2 [11, 3.4.9, 3.4.16]. *There is a set of maps J in Sp^{Σ} such that any stable trivial cofibration is a retract of a sequential colimit of pushouts of maps in J.*

This provides a general method for proving that the class of cofibrations or trivial cofibrations has some property. One shows that the generating maps have some property and that the property is preserved under pushouts, colimits, and retracts. This method is central to the proofs in Sections 3 and 4.

Quillen's small object argument [14, p. II 3.4] has an analogue which allows one to functorially factor maps whenever the model category is cofibrantly generated, see [11, 3.2.11].

DEFINITION 2.1.3. Let L be the functorial *stable fibrant replacement* functor defined by functorially factoring the map $X \longrightarrow *$ into a stable trivial cofibration, $X \longrightarrow LX$ and a stable fibration $LX \longrightarrow *$. This is the factorization one defines using the small object argument applied to the set of maps J. Using J', the generators of the level trivial cofibrations, instead, one defines L' as the functorial *level fibrant replacement* with $X \to L'X$ a level trivial cofibration and $L'X \to *$ a level fibration.

Semistable objects and π_*-isomorphisms

Comparing symmetric spectra to the model category of spectra, $Sp^{\mathbf{N}}$, defined in [2] sheds light on the complications involved in the model category of symmetric spectra. There is a forgetful functor $U\colon Sp^{\Sigma} \longrightarrow Sp^{\mathbf{N}}$ which forgets the action of the symmetric groups and uses the structure maps $S^1 \wedge X_n \longrightarrow X_{1+n}$.

DEFINITION 2.1.4. Let $\pi_k(X) = \pi_k(UX) = \operatorname{colim}_i \pi_{k+i} X_i$. A map f of symmetric spectra is a π_*-*isomorphism* if it induces an isomorphism on these *classical stable homotopy groups*.

These classical stable homotopy groups are *not* the maps in the homotopy category of symmetric spectra of the sphere into X. For example, $\lambda\colon F_1 S^1 \longrightarrow F_0 S^0$, adjoint to the identity map $S^1 \to Ev_1(F_0 S^0)$, is a stable equivalence but it is not a π_*-isomorphism. As shown in [11, 3.1.11], though, a π_*-isomorphism is a particular example of a stable equivalence. Hence, to avoid confusion, we use the term π_*-isomorphism instead of stable homotopy isomorphism and call these the classical stable homotopy groups instead of just stable homotopy groups. In Section 3 we construct a functor, D, which converts any stable equivalence into a π_*-isomorphism between semistable spectra, see Definition 2.1.6 below.

As in [2], we define a functor Q for symmetric spectra.

DEFINITION 2.1.5. Define $QX = \operatorname{colim}_n \Omega^n L' \operatorname{sh}_n X$.

This functor does not have the same properties as in [2]. For instance, QX is not always an Ω-spectrum and $X \to QX$ is not always a π_*-isomorphism. One property that does continue to hold, however, is that a map f is a π_*-isomorphism if and only if Qf is a level equivalence. Also, QX is always level fibrant.

DEFINITION 2.1.6. A *semistable* symmetric spectrum is one for which the stable fibrant replacement map, $X \to LX$, is a π_*-isomorphism.

Of course $X \to LX$ is always a stable equivalence, but not all spectra are semistable. For instance, $F_1 S^1$ is not semistable. Any stably fibrant spectrum, i.e., an Ω-spectrum, is semistable though. The following proposition shows that on semistable spectra Q has the same properties as in [2] on $Sp^{\mathbf{N}}$.

PROPOSITION 2.1.7. *The following are equivalent.*

(1) *The symmetric spectrum X is semistable.*
(2) *The map $X \to \Omega L' \operatorname{sh}_1 X$ is a π_*-isomorphism.*
(3) *$X \to QX$ is a π_*-isomorphism.*
(4) *QX is an Ω-spectrum.*

This proposition, [11, 5.6.2] with R_∞ there, is replaced by Q here.

Two classes of semistable spectra are described in the following proposition. The second class includes the connective and convergent spectra.

PROPOSITION 2.1.8 [11, 5.6.4].

(1) *If the classical stable homotopy groups of X are all finite then X is semistable.*
(2) *Suppose that X is a level fibrant symmetric spectrum and there exists some $\alpha > 1$ such that $X_n \to \Omega X_{n+1}$ induces an isomorphism $\pi_k X_n \to \pi_{k+1} X_{n+1}$ for all $k \leqslant \alpha n$ for sufficiently large n. Then X is semistable.*

The next proposition shows that stable equivalences between semistable spectra are particularly easy to understand.

PROPOSITION 2.1.9 [11, 5.6.5]. *Let $f\colon X \to Y$ be a map between two semistable symmetric spectra. Then f is a stable equivalence if and only if it is a π_*-isomorphism.*

Finally, any spectrum π_*-isomorphic to a semistable spectrum is itself semistable.

PROPOSITION 2.1.10. *If $f\colon X \to Y$ is a π_*-isomorphism and Y is semistable then X is semistable.*

Proof. Since Lf and $Y \to LY$ are π_*-isomorphisms, $X \to LX$ is also a π_*-isomorphism. □

Symmetric ring spectra

In this section, rings, modules, and algebras are defined for symmetric spectra.

DEFINITION 2.1.11. A *symmetric ring spectrum* is a monoid in the category of symmetric spectra. In other words, a symmetric ring spectrum is a symmetric spectrum, R, with maps $\mu\colon R \wedge_S R \to R$ and $\eta\colon S \to R$ such that they are associative and unital, i.e., $\mu\circ(\mu \wedge_S \mathrm{id}) = \mu\circ(\mathrm{id} \wedge_S \mu)$ and $\mu\circ(\eta \wedge_S \mathrm{id}) \cong \mathrm{id} \cong \mu\circ(\mathrm{id} \wedge_S \eta)$. R is called *commutative* if $\mu \circ \mathrm{tw} = \mu$, where $\mathrm{tw}\colon R \wedge_S R \to R \wedge_S R$ is the twist isomorphism.

Since symmetric ring spectra are the only type of ring spectra in this paper we also refer to them as simply *ring spectra*.

Remark 2.1.12. This description of a symmetric ring spectrum agrees with the definition of a functor with smash product defined on spheres as in [10, 2.7]. The centrality condition mentioned in [10, 2.7.ii] is necessary but was not included in some earlier definitions of FSPs defined on spheres. Note, however, that there are no connectivity (e.g. $F(S^{n+1})$ is n-connected) or convergence conditions (e.g. the limit is attained at a finite stage in the colimit defining π_n for each n) placed on symmetric ring spectra. These conditions are usually assumed although not always explicitly stated when using FSPs. In particular, these conditions are necessary for applying Bökstedt's approximation theorem [1, 1.6]. Corollary 3.1.7 shows that a special case of this approximation theorem holds for any semistable spectrum. To consider non-convergent spectra we use Theorem 3.1.2 in place of the approximation theorem. This theorem does not require any connectivity or convergence conditions.

Proposition 2.1.8 shows that the connectivity and convergence conditions on an FSP ensure that the associated underlying symmetric spectrum is semistable. Proposition 2.1.9 shows that stable equivalences between such FSPs are exactly the π_*-isomorphisms. As with the category of symmetric spectra, inverting the π_*-isomorphisms is not enough to ensure that the homotopy category of symmetric ring spectra is equivalent to the homotopy category of A_∞-ring spectra. So once the connectivity and convergence conditions are removed one must consider stable equivalences instead of just π_*-isomorphisms.

We also need the following definitions of R-modules and R-algebras in later sections.

DEFINITION 2.1.13. Let R be a symmetric ring spectrum. A (left) R-*module* is a symmetric spectrum M with a map $\alpha\colon R \wedge_S M \to M$ that is associative and unital.

DEFINITION 2.1.14. Let R be a commutative ring spectrum. An *R-algebra* is a monoid in the category of R-modules. That is, an R-algebra is a symmetric spectrum A with R-module maps $\mu\colon A \wedge_R A \to A$ and $R \to A$ that satisfy the usual associativity and unity diagrams.

Note that symmetric ring spectra are exactly the S-algebras. The following lemma is needed for Section 4 so we reproduce it for the reader's convenience.

LEMMA 2.1.15 [11, 5.4.4]. *Let R be a symmetric ring spectrum and M a cofibrant R-module. Then $M \wedge_R -$ takes level equivalences of R-modules to level equivalences in Sp^Σ and it takes stable equivalences of R-modules to stable equivalences in Sp^Σ.*

2.2. HOMOTOPY COLIMITS

In this section we list some of the properties of the homotopy colimit functor for symmetric spectra which are used in the latter parts of this paper. The most important property is that the homotopy colimit of symmetric spectra can be defined by using the homotopy colimit of spaces at each level, see Definition 2.2.1. We use the basic construction of the homotopy colimit for spaces from [3].

DEFINITION 2.2.1. Let B be a small category and $F\colon B \to Sp^\Sigma$ a diagram of symmetric spectra. Let F_l denote the diagram of spaces at level l. Then $(\mathrm{hocolim}^B_{Sp^\Sigma} F)_l = \mathrm{hocolim}^B_{\mathcal{S}_*} F_l$.

This definition makes sense because any stable cofibration is a level cofibration and colimits in Sp^Σ are created on each level. Also, this homotopy colimit has the usual properties of a homotopy colimit. Namely, a map between diagrams which is objectwise a level equivalence, a π_*-isomorphism, or a stable equivalence induces the same type of equivalence on the homotopy colimit. The next two propositions consider the first two cases. The case of stable cofibrations could be proved by generalizing [3, XII 4.2] to arbitrary model categories. Instead, here we use the detection functor developed in Section 3 to verify this property in Lemma 4.1.5.

PROPOSITION 2.2.2. *Let $F, G\colon B \to Sp^\Sigma$ be two diagrams of symmetric spectra with a natural transformation $\eta : F \to G$ between them. If $\eta(b)\colon F(b) \to G(b)$ is a level equivalence at each object $b \in B$, then* $\mathrm{hocolim}^B F \to \mathrm{hocolim}^B G$ *is a level equivalence.*

Proof. This follows from the dual of [3, XI 5.6]. Cofibrancy conditions are not required here since any space (i.e., simplicial set) is cofibrant. □

Following [3, XII 5] there is a spectral sequence for calculating any homology theory applied to the homotopy colimit of spaces. The spectral sequence is associated to the filtration of the homotopy colimit given by the length of the sequence of maps in B. So for $F: B \to \mathcal{S}_*$ this spectral sequence converges to h_* hocolim^B F and has E^2-term $E^2_{s,t} = \operatorname{colim}^s_B(h_t F)$.

We use the following lemma to go from the homology theory π^s_* defined on spaces by $\pi^s_* K = \pi_* F_0 K$ to one on Sp^Σ.

LEMMA 2.2.3. *For X a symmetric spectrum, $\pi_* X = \operatorname{colim}_n \pi^s_* X_n$.*

Proof. Consider the lattice of spaces $\Omega^i(\Omega^j L'\Sigma^j X_i)$ indexed over $(i, j) \in \mathbb{N}\times\mathbb{N}$ with maps for fixed j using the adjoint structure maps of $\Omega^j L'\Sigma^j X$ and for fixed i using $\Omega^i\Omega^j$ applied to the adjoint structure maps of $L'F_0 X_i$. Applying homotopy and taking colimits in the two different directions finishes the proof. In one direction, one gets $\operatorname{colim}_j \pi_*\Omega^j L'\Sigma^j X$, but each of these terms and hence the colimit is isomorphic to $\pi_* X$. In the other direction, one has $\operatorname{colim}_i \pi^s_* X_i$. $\square$

So applying the homology theory π_*, the above spectral sequence calculates π_* of each level of the homotopy colimit. Since $\pi_* X = \operatorname{colim}_n \pi^s_* X_n$ and a sequential colimit of spectral sequences is a spectral sequence, taking the colimit of these level spectral sequences produces a spectral sequence.

PROPOSITION 2.2.4. *For $F: B \to Sp^\Sigma$, there is a spectral sequence converging to π_*hocolim$^B_{Sp^\Sigma}$ F with E^2-term $E^2_{s,t} = \operatorname{colim}^s_B(\pi_t F)$.*

This spectral sequence shows that homotopy colimits preserve objectwise π_*-isomorphisms.

PROPOSITION 2.2.5. *Let $F, G: B \to Sp^\Sigma$ be two diagrams of symmetric spectra with a natural transformation $\eta: F \to G$ between them. If $\eta(b): F(b) \to G(b)$ induces a π_*-isomorphism at each object $b \in B$ then* hocolim^B $F \to$ hocolim^B G *induces a π_*-isomorphism.*

Proof. Since η induces a π_*-isomorphism between the two diagrams in question, it induces an E^2-isomorphism. Thus, it induces an isomorphism on the E^∞-term, and hence, a π_*-isomorphism on the homotopy colimits.

In Section 3, we consider diagrams over the skeleton of the category of finite sets and injections, I, with objects $\mathbf{n}$. Let I_m denote the full subcategory of I whose objects are $\mathbf{n}$ where n is greater than or equal to m. The following lemma states the cofinality information relating these categories.

LEMMA 2.2.6. *Let $F: I \to Sp^\Sigma$ be a diagram of spectra. The inclusion u_m: $I_m \to I$ is terminal, hence* hocolim^{I_m} $u^*_m F \to$ hocolim^I F *is a level equivalence.*

Proof. Consider the functor $- + m: I \to I_m$ which induces a functor on any under category. There is a natural transformation from the identity functor to both $u_m \circ (- + m)$ and $(- + m) \circ u_m$. Hence each under category is homotopy equivalent to $(i \downarrow I)$. But $(i \downarrow I)$ is contractible because it has an initial object $1: i \to i$. The homotopy colimit statements follow from [3, XI 9.2]. $\square$

Using this cofinality result we prove the following proposition.

PROPOSITION 2.2.7. *Let* $F, G\colon I \to \mathcal{S}_*$ *be two diagrams of spaces with a natural transformation* $\eta\colon F \to G$ *between them. Assume that* $\eta(\mathbf{n})\colon F(\mathbf{n}) \to G(\mathbf{n})$ *is a* $\lambda(n)$ *connected map, where* $\lambda(n) \leqslant \lambda(n+1)$ *and* $\lim_n \lambda(n)$ *is infinite. Then* $\mathrm{hocolim}^I F \to \mathrm{hocolim}^I G$ *is a weak equivalence.*

Proof. The map is an N-equivalence for every $N > 0$. Choose an n such that $\lambda(n) > N$. Then for every object $\mathbf{m}$ in I_n the map $\eta\colon F(\mathbf{m}) \to G(\mathbf{n})$ is an N-equivalence, and so we conclude that $\eta\colon \mathrm{hocolim}^{I_n} u_n^* F \to \mathrm{hocolim}^{I_n} u_n^* G$ is an N-equivalence. The proposition follows by Lemma 2.2.6. □

We also need the following proposition which shows that the homotopy colimit of a diagram of level equivalences over I is level equivalent to its value at $\mathbf{0}$.

PROPOSITION 2.2.8. *Let* $F\colon I \to Sp^\Sigma$ *be a diagram of spectra. Assume that for each morphism* f *in* I, $F(f)$ *is a level equivalence. Then the inclusion* $F(\mathbf{0}) \to \mathrm{hocolim}^I F$ *is a level equivalence.*

Proof. Consider the constant functor $C\colon I \to Sp^\Sigma$ with constant value $F(\mathbf{0})$. Then at each object the map $C(\mathbf{n}) = F(\mathbf{0}) \to F(\mathbf{n})$ induced by the unique map $\mathbf{0} \to \mathbf{n}$ is a level equivalence. Hence, by Proposition 2.2.2, it induces a level equivalence on the homotopy colimits, $F(\mathbf{0}) \to \mathrm{hocolim}^I F$. □

Finally, we need the following proposition due to Jeff Smith, [18]. Let T be the category with objects $\mathbf{n} = \{1, \dots, n\}$ and morphisms the standard inclusions. Homotopy colimits over T are weakly equivalent to telescopes. Let ω be the ordered set of natural numbers and I_ω be the category whose objects are the finite sets $\mathbf{n}$ and the set ω and whose morphisms are inclusions. Let $L_h F\colon I_\omega \to \mathcal{S}_*$ be the left homotopy Kan extension of $F\colon I \to \mathcal{S}_*$ along the inclusion of categories $i\colon I \to I_\omega$.

PROPOSITION 2.2.9. *Let* M *be the monoid of injective maps* $i\colon \omega \to \omega$ *under composition. Given any functor* $F\colon I \to \mathcal{S}_*$, *then*

(1) $\mathrm{hocolim}^I F$ *is weakly equivalent to* $(L_h F(\omega))_{hM}$ *where* $(-)_{hM}$ *is the homotopy orbits with respect to the action of* M, *and*
(2) $L_h F(\omega)$ *is weakly equivalent to* $\mathrm{hocolim}^T F$.

Proof. For the convenience of the reader we sketch Smith's proof of this proposition. Since $L_h F$ is the homotopy Kan extension, $\mathrm{hocolim}^I F \simeq \mathrm{hocolim}^{I_\omega} L_h F$. Next, consider the full subcategory, A of I_ω with just one object, ω. Since the inclusion of A in I_ω is terminal, $\mathrm{hocolim}^{I_\omega} L_h F$ is weakly equivalent to $\mathrm{hocolim}^A L_h F$, by [3, XI 9.2]. Since $\mathrm{Hom}_A(\omega, \omega) = M$, $\mathrm{hocolim}^A L_h F$ is the homotopy orbit space $(L_h F(\omega))_{hM}$.

For the second statement, $L_h F(\omega) = \text{hocolim}^{i(n)\to\omega\in(i\downarrow\omega)} F(n)$. Here i is the inclusion $I \to I_\omega$. The category T described above is equivalent to the category $(i \circ \alpha \downarrow \omega)$ for the inclusion $\alpha\colon T \to I$. This category $(i \circ \alpha \downarrow \omega)$ is terminal in $(i \downarrow \omega)$, because every under category has an initial object. So by [3, XI 9.2], $L_h F(\omega)$ is weakly equivalent to $\text{hocolim}^T F$. □

3. Detecting Stable Equivalences

In this section we introduce a functor, D, which detects stable equivalences in the sense that a map $X \to Y$ is a stable equivalence if and only if $DX \to DY$ is a π_*-isomorphism. Of course the stable fibrant replacement functor L, 2.1.3, also has this property. It even turns stable equivalences into level equivalences. The drawback of L is that its only description is via the small object argument. Hence it is difficult to say much about L apart from its abstract properties. The advantage of the functor D is that it has a more explicit definition. In particular, there is a spectral sequence for calculating the classical stable homotopy groups of DX, see Proposition 2.2.4. Moreover, these groups are invariants of the stable equivalence type of X because D takes stable equivalences to π_*-isomorphisms.

In Section 4 we see that D fits into a sequence of functors used to define THH in [1]. We use the notation D instead of THH_0 because D is defined on any symmetric spectrum, not just on ring spectra.

3.1. MAIN STATEMENTS AND PROOFS

The detection functor D is a homotopy colimit over the diagram category I, the skeleton of finite sets and injections with objects $\mathbf{n}$. Given a symmetric spectrum X, define a functor $\mathcal{D}_X\colon I \to Sp^\Sigma$ whose value on the object $\mathbf{n}$ is $\Omega^n L' F_0 X_n$. Recall L' is just a level fibrant replacement functor. For a standard inclusion of a subset $\alpha\colon \mathbf{n} \subset \mathbf{m}$ the map $\mathcal{D}_X(\alpha)$ is just $\Omega^n L'$ applied to the composition of maps $F_0 X_n \to F_0 \Omega^{m-n} X_m \to \Omega^{m-n} F_0 X_m$ induced by the structure maps of X. For an isomorphism, the action is given by the conjugation action on the loop coordinates and on X_n. All morphisms in I are compositions of isomorphisms and these standard inclusions.

DEFINITION 3.1.1. The *detection functor* $D\colon Sp^\Sigma \to Sp^\Sigma$ is defined by

$$DX = \text{hocolim}^I_{Sp^\Sigma} \mathcal{D}_X.$$

The homotopy colimit of symmetric spectra is given by a level homotopy colimit of spaces, see 2.2.1. Hence

$$(DX)_n = \text{hocolim}^{k\in I}_{\mathcal{S}_*} \Omega^k L' \Sigma^n X_k.$$

The next theorem states that D detects stable equivalences.

THEOREM 3.1.2. *The following are equivalent.*

(1) *$X \to Y$ is a stable equivalence.*
(2) *$DX \to DY$ induces a π_*-isomorphism.*
(3) *$D^2X \to D^2Y$ is a level equivalence.*
(4) *$QDX \to QDY$ is a level equivalence.*

Remark 3.1.3. One can apply the forgetful functor $U\colon\ Sp^{\Sigma} \to Sp^{\mathbf{N}}$ after applying D. Then, although the forgetful functor does not detect and preserve stable equivalences, the composition of this detection functor with the forgetful functor does detect and preserve weak equivalences.

Remark 3.1.4. One could consider symmetric spectra over topological spaces instead of simplicial sets here. Theorem 3.1.2 and all of the statements leading up to it in this section and in Section 2.2 which do not involve the functor Q hold when the objects involved are levelwise non-degenerately based spaces. Hence, D also detects stable equivalences between symmetric spectra based on topological spaces. More precisely, let c be a cofibrant replacement functor of spaces applied levelwise, then $X \to Y$ is a stable equivalence if and only if $DcX \to DcY$ is a π_*-isomorphism.

To modify these statements for topological spaces, note that homotopy colimits of non-degenerately based spaces are invariant under weak homotopy equivalences. For the statements involving Q one needs stably cofibrant symmetric spectra because homotopy groups must commute with sequential colimits. But these statements are separate from those involving D.

Theorem 3.1.2 considers the properties of D with respect to morphisms. The following theorem considers the properties of D on objects.

THEOREM 3.1.5. *Let X be a symmetric spectrum.*

(1) *DX is semistable.*
(2) *If X is semistable, then the level fibrant replacement of DX, $L'DX$, is an Ω-spectrum.*

Since stable equivalences between semistable spectra are π_*-isomorphisms and between Ω-spectra are level equivalences, Theorem 3.1.5 shows that the second and third statements of Theorem 3.1.2 really just say that D and D^2 preserve and detect stable equivalences.

Theorem 3.1.2 shows that the classical stable homotopy groups of DX are a stable equivalence invariant. In the next theorem we show that they are in fact the derived classical stable homotopy groups, i.e., they are isomorphic to π_*LX.

THEOREM 3.1.6. *Let X be a symmetric spectrum.*

(1) *There is a natural zig-zag of functors inducing π_*-isomorphisms between LX and DX.*

(2) *There are natural zig-zags of functors inducing level equivalences between LX, D^2X, and QDX.*

This theorem shows that the fibrant replacement functor is determined up to π_*-isomorphism by D or up to level equivalence by D^2 or QD. The spectral sequence for calculating the classical stable homotopy groups of DX, Proposition 2.2.4, thus calculates the derived stable homotopy groups $\pi_* DX \cong \pi_* LX$.

COROLLARY 3.1.7. *For X any semistable spectrum, X and DX are π_*-isomorphic. Moreover, QX and DX are level equivalent.*

LX and QX are level equivalent for X semistable, so the second statement follows from Proposition 3.1.9(3) below.

Remark 3.1.8. This corollary is a special case of [1, 1.6] where the convergence and connectivity conditions are replaced by the semistable condition. By Proposition 2.1.8 we recover a statement with convergence conditions but no connectivity conditions. But this corollary also applies for instance when the classical stable homotopy groups of X are all finite, by Proposition 2.1.8.

The proofs of Theorems 3.1.2 and 3.1.5 use the following properties of the functor D.

PROPOSITION 3.1.9. *Let $f\colon X \to Y$ be a map of symmetric spectra.*

(1) *If f is a stable equivalence then Df is a π_*-isomorphism.*
(2) *If f is a π_*-isomorphism then Df is a level equivalence.*
(3) *For any semistable spectrum X, there is a natural zig-zag of functors inducing level equivalences between LX and DX.*

We assume Proposition 3.1.9 to prove Theorems 3.1.2, 3.1.5, and 3.1.6. The proof of Proposition 3.1.9 is technical, so it is delayed until the next subsection.

Proof of Theorem 3.1.6. By Proposition 3.1.9 (3) applied to LX there is a zig-zag of level equivalences between LLX and DLX. By Proposition 3.1.9 (1) since $X \to LX$ is a stable equivalence $DX \to DLX$ is a π_*-isomorphism. Putting these equivalences together with the fact that LLX is level equivalent to LX, we get a zig-zag of π_*-isomorphisms between LX and DX.

Applying D to the zig-zag of π_*-isomorphisms between LX and DX shows that DLX and D^2X are level equivalent by Proposition 3.1.9 (2). Combining this with the zig-zag of level equivalences between LX and DLX produces the level equivalence of LX and D^2X. The equivalences for QDX are similar. □

Proof of Theorem 3.1.5. By Theorem 3.1.6 DX is π_*-isomorphic to LX. LX is an Ω-spectrum, hence it is semistable. So by Proposition 2.1.10, DX is semistable.

For X semistable, Proposition 3.1.9 shows that DX is level equivalent to LX, an Ω-spectrum. Hence $L'DX$ is an Ω-spectrum. □

Proof of Theorem 3.1.2. Proposition 3.1.9 shows that (1) implies (2) and (2) implies (3). A map f is a π_*-isomorphism if and only if Qf is a level equivalence. Hence the second and fourth statements are also equivalent.

By Theorem 3.1.6 part 2, LX and D^2X are naturally level equivalent. Hence if $D^2X \to D^2Y$ is a level equivalence then so is $LX \to LY$. But this is equivalent to $X \to Y$ being a stable equivalence. □

3.2. PROOF OF PROPOSITION 3.1.9

The proof of Proposition 3.1.9 is more technical. In this subsection we first prove the second part of Proposition 3.1.9. Using this we prove the third part. Then, for the first part of Proposition 3.1.9 we state and prove several lemmas which together finish the proof. The proof of the first part is the most technical and heavily uses model category techniques. Throughout this section we use several of the properties of the homotopy colimit developed in Section 2.2.

Proof of Proposition 3.1.9 Part 2. We apply Lemma 2.2.9, due to Jeff Smith, to each level of D. Consider the zeroth level first. If f is a π_*-isomorphism then $\text{hocolim}^T\, \Omega^n L' f_n$ is a weak equivalence, since $\pi_* X = \pi_* \text{hocolim}^T\, \Omega^n L' X_n$. Since taking homotopy orbits preserves weak equivalences this shows that the zeroth level of $DX \to DY$ is a weak equivalence, i.e., $\text{hocolim}^I\, \Omega^n L' f_n$ is a weak equivalence.

The kth level of DX is the 0th level of $D\Sigma^k X$. Since $\Sigma^k f$ is a π_*-isomorphism if f is, this shows that each level is a weak equivalence. □

Recall that $(sh_n X)_k = X_{n+k}$, [11, 2.2.12] and L' is a level fibrant replacement functor.

DEFINITION 3.2.1. Define $MX = \text{hocolim}^I\, \Omega^n L' sh_n X$.

Proof of Proposition 3.1.9 Part 3. First we develop the transformations which play a part in the zig-zag mentioned in the proposition. The inclusion of the object **0** in I induces a natural map $X \longrightarrow MX$. There is also a natural transformation of functors $D \longrightarrow M$. The structure maps on X induce a natural map of symmetric spectra $F_0 X_n \longrightarrow sh_n X$. Applying $\Omega^n L'$ to this map induces a map of diagrams over I, and hence a natural map of homotopy colimits. So there is a natural zig-zag $X \longrightarrow MX \longleftarrow DX$. The zig-zag mentioned in the proposition is this zig-zag applied to LX along with the natural map $DX \to DLX$.

For semistable X, the map $X \to LX$ is a π_*-isomorphism. So $DX \to DLX$ is a level equivalence by Proposition 3.1.9 part 2. So we show that if X is an Ω-spectrum, then both of the maps $X \to MX \leftarrow DX$ are level equivalences.

By definition an Ω-spectrum is a level fibrant spectrum such that $X \to \Omega\,\mathrm{sh}_1\,X$ is a level equivalence. Since both shift and Ω preserve level equivalences (on level fibrant spectra), each map in the diagram over I used to define MX is a level equivalence. By Proposition 2.2.8, then $X \to MX$ is a level equivalence.

To show $DX \to MX$ is a level equivalence for any Ω-spectrum X, we consider connective covers. Given a level fibrant spectrum X define its kth connective cover, C_kX, as the homotopy fiber of the map from X to its kth Postnikov stage P_kX. The kth Postnikov functor is the localization functor given by localizing with respect to the set of maps $\{F_n\partial\Delta[m+n+k+2] \to F_n\Delta[m+n+k+2]\colon\ m, n \geqslant 0\}$. At level n, this functor is weakly equivalent to the $(n+k)$th Postnikov functor on spaces which is given by localization with respect to the set of maps $\{\partial\Delta[m+n+k+2] \to \Delta[m+n+k+2]\colon\ m \geqslant 0\}$ (see also [8]). Then $(C_kT)_n$ is $n+k$ connected and $\pi_i(C_kT)_n \to \pi_iT_n$ is an isomorphism for $i > n+k$. Note that any level fibrant spectrum is level equivalent to the homotopy colimit over its connective covers. As $-k$ decreases, the homotopy type of each level of $C_{-k}X$ eventually becomes constant. So $\mathrm{hocolim}_k\, C_{-k}X \to X$ is a level equivalence.

Because Ω^m, L' and F_0 commute up to level equivalence with sequential homotopy colimits and homotopy colimits commute, $\mathrm{hocolim}_n\, DC_{-n}X$ is level equivalent to DX. The shift functor also commutes with homotopy colimits so $\mathrm{hocolim}_n\, MC_{-n}X$ is level equivalent to MX. So, to apply Proposition 2.2.2, we need to show DC_nX and MC_nX are level equivalent for each n.

In the diagrams creating these homotopy colimits, consider level l at the object $\mathbf{m}$ in I. The map in question is $(\Omega^m L'\Sigma^l C_nX)_m \to (\Omega^m L' C_nX)_{m+l}$. In general the map $\Sigma^l\Omega^l Y \to Y$ is $2N-l+1$ connected when Y is N connected. Hence the map in question is $2n+m+l+1$ connected. Using Proposition 2.2.7, this connectivity implies that $(DC_nX)_k \to (MC_nX)_k$ is a weak equivalence. □

The proof of Proposition 3.1.9 part 1 breaks up into several parts by using model category theory techniques. Since any stable equivalence can be factored as a stable trivial cofibration followed by a level trivial fibration, we show that D takes both stable trivial cofibrations and level equivalences to π_*-isomorphisms. For the case of stable trivial cofibrations we split the problem further into showing that D of any generating stable trivial cofibration is a π_*-isomorphism and that D behaves well with respect to push outs, i.e., that the following two lemmas hold. Since any stable trivial cofibration is a retract of a sequential colimit of pushouts of generating trivial cofibrations by 2.1.2, these lemmas suffice to finish the proof.

LEMMA 3.2.2. *Let $j\colon A \to B$ be a generating stable trivial cofibration. Then $Dj : DA \to DB$ is a π_*-isomorphism.*

LEMMA 3.2.3. *If*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is a pushout square with $A \to B$ a cofibration, then

$$\begin{array}{ccc} DA & \longrightarrow & DX \\ \downarrow & & \downarrow \\ DB & \longrightarrow & DY \end{array}$$

is a homotopy pushout square. That is, if P is the homotopy colimit of $DB \leftarrow DA \to DX$, then $P \to DY$ is a stable equivalence. In fact, $P \to DY$ is a π_*-isomorphism.

Combining this lemma with the next shows that if $DA \to DB$ is a π_*-isomorphism then $DX \to DY$ is also a π_*-isomorphism.

LEMMA 3.2.4. *Let*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

be a square in Sp^Σ with Y π_*-isomorphic to the homotopy pushout. Assume $A \to B$ is a π_*-isomorphism. Then $X \to Y$ is a π_*-isomorphism.

For a proper model category this is a standard fact, that the homotopy pushout of a weak equivalence is a weak equivalence. But no model category on symmetric spectra has been written down with weak equivalences the π_*-isomorphisms, so we prove this below.

Proof of Proposition 3.1.9 Part 1. Assuming Lemmas 3.2.2, 3.2.3, and 3.2.4, we can finish this proof. As mentioned above, we factor a stable equivalence into a stable trivial cofibration followed by a level trivial fibration and show D takes both pieces to π_*-isomorphisms. A level equivalence induces a level equivalence at each object in the diagram for defining D. Hence, by Proposition 2.2.2, D of a level equivalence is a level equivalence, and thus a π_*-isomorphism.

Next we build an arbitrary stable trivial cofibration as a retract of a sequential colimit of pushouts of generating cofibrations by 2.1.2. Since retracts and sequential colimits preserve π_*-isomorphisms we only need to consider pushouts of generating stable trivial cofibrations. By Lemma 3.2.2, D of a generating trivial cofibration is a π_*-isomorphism. Then, by Lemmas 3.2.3 and 3.2.4, D of any map formed by a pushout of a generating stable trivial cofibration is a π_*-isomorphism. □

The rest of this section is devoted to proving these three lemmas.

Proof of Lemma 3.2.4. Factor the map $A \to X$ as a stable cofibration followed by a level trivial fibration $A \to Z \to X$. Then form the pushout square as follows:

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & & \downarrow \\ B & \longrightarrow & P. \end{array}$$

Since the top map is a level cofibration, P is the homotopy pushout of this square. Since $A \to B$ is a π_*-isomorphism, $Z \to P$ is a π_*-isomorphism because π_* is a homology theory. Since $Z \to X$ is a level equivalence, to see that $X \to Y$ is a π_*-isomorphism it is enough to know that $P \to Y$ is a π_*-isomorphism. But this is assumed as part of the hypotheses. □

Proof of Lemma 3.2.3. $P \to DY$ is a π_*-isomorphism, because homotopy colimits commute. Let P^n be the homotopy pushout at the object $\mathbf{n} \in I$ of $\mathcal{D}_B \leftarrow \mathcal{D}_A \to \mathcal{D}_X$. Then P is level equivalent to $\text{hocolim}^I P^n$. Proposition 2.2.5 shows that a map of diagrams which is a π_*-isomorphism at each object induces a π_*-isomorphism on the homotopy colimits. Hence, it is enough to show that $P^n \to \Omega^n L' F_0(Y_n)$ is a π_*-isomorphism for each n.

Since cofibrations induce level cofibrations and F_0 preserves cofibrations and pushouts, F_0 applied to each level of the pushout square in the lemma is a homotopy pushout square. Since $X \to L'X$ is a level equivalence it preserves homotopy pushout squares up to level equivalence. Since Ω^n only shifts π_* by n, it preserves homotopy pushouts up to π_*-isomorphism. Hence $P^n \to \Omega^n L' F_0(Y_n)$ is a π_*-isomorphism. □

We are left with proving Lemma 3.2.2. This proof goes to the heart of why D detects stable equivalences. Basically this is because the free symmetric spectra, $F_n K$, are closely related to free diagrams over I, see 2.1.1. The following lemma and its proof make this statement more exact and identify the stable homotopy type of $DF_m(K)$.

LEMMA 3.2.5. *There is a* $2l - m - 1$ *connected map* $\psi_l \colon \Omega^m L'(S^l \wedge K) \to (DF_m K)_l$. *These maps fit together to give a map of symmetric spectra* $\psi\colon \Omega^m L' F_0 K \to DF_m K$ *which is a* π_**-isomorphism.*

To prove this lemma we define free diagrams on the category I.

DEFINITION 3.2.6. Define $\mathcal{F}_m K \colon I \to \mathcal{S}_*$ by $(\mathcal{F}_m K)(n) = \text{hom}_I(m, n)_+ \wedge K$.

$\mathcal{F}_m(-)$ is left adjoint to the functor from I-diagrams over $\mathcal{S}_*$ to $\mathcal{S}_*$ which evaluates the diagram at $m \in I$. Hence a natural transformation from $\mathcal{F}_m K$ into any diagram over I is determined by a map from K to the diagram evaluated at m.

Proof of Lemma 3.2.5. Let $\mathcal{D}^l_{F_m K}\colon\ I \to \mathcal{S}_*$ be the functor given by the lth level of the functor $\mathcal{D}_{F_m K}$. Then there is a map $\phi_l\colon\ \mathcal{F}_m \Omega^m L'(S^l \wedge K) \to \mathcal{D}^l_{F_m K}$ determined by the inclusion of the wedge summand corresponding to the identity map, $\Omega^m L'(S^l \wedge K) \to \Omega^m L'(S^l \wedge \hom_I(m,m)_+ \wedge K)$. As the homotopy colimit of a free diagram is weakly equivalent to the colimit (see [7, 15]), the homotopy colimit of this map is the map $\psi_l\colon\ \Omega^m L'(S^l \wedge K) \to (DF_m K)_l$ mentioned in the lemma.

The map of diagrams is $2l-m-1$ connected at each spot. At each $n \in I$, $\phi_l(n)$, factors into two maps as follows,

$$\begin{aligned}&\hom_I(m,n)_+ \wedge \Omega^m L'(S^l \wedge K) \to \Omega^m L'(\hom_I(m,n)_+ \wedge S^l \wedge K)\\ &\quad \to \Omega^m \Omega^{n-m} L' \Sigma^{n-m}(\hom_I(m,n)_+ \wedge S^l \wedge K).\end{aligned}$$

The first map is $2l-m-1$ connected by the Blakers–Massey theorem which shows that a wedge of loop spaces, $\Omega X \vee \Omega Y$, is equivalent in the stable range to the loop of the wedge, $\Omega(X \vee Y)$. The second map is $2l-m-1$ connected by the Freudenthal suspension theorem, which for simplicial sets concerns the map $X \to \Omega L' \Sigma X$. Hence the map at each spot in the diagram, $\phi_l(n)$ and thus the map of homotopy colimits, ψ_l is $2l-m-1$ connected.

To see that these levels fit together, note that we can prolong $\mathcal{F}_m$ to a functor from symmetric spectra to I-diagrams of symmetric spectra. Then there is a map $\phi\colon\ \mathcal{F}_m(\Omega^m L' F_0 K) \to \mathcal{D}_{F_m K}$ which on level l is given by the map, ϕ_l, above. Hence, taking homotopy colimits, this induces a map $\psi\colon\ \Omega^m L' F_0 K \to DF_m K$ which is a π_*-isomorphism. □

Proof of Lemma 3.2.2. Some of the generating trivial cofibrations are in fact level equivalences, $F_n(\Delta^l[k]_+) \to F_n(\Delta[k]_+)$. But, D of a level equivalence is a level equivalence. Recall from [11, 3.4.9], the other generating trivial cofibrations are the maps $P(c_m, j_r)\colon\ P_{m,r} \to C\lambda_m \wedge_S F_0(\Delta[r]_+)$ where $P_{m,r}$ is the pushout below.

$$\begin{array}{ccc} F_{m+1}(S^1 \wedge \dot{\Delta}[r]_+) & \longrightarrow & F_{m+1}(S^1 \wedge \Delta[r]_+) \\ \big\downarrow & & \big\downarrow \\ C\lambda_m \wedge_S F_0(\dot{\Delta}[r]_+) & \longrightarrow & P_{m,r} \end{array}$$

To show that D of $P(c_m, j_r)$ is a π_*-isomorphism, it is only necessary to show that D of $c_K\colon\ F_{m+1}(S^1 \wedge K) \to C\lambda_m \wedge_S F_0 K$ is a π_*-isomorphism for $K = \dot{\Delta}[r]_+$ or $\Delta[r]_+$. This is enough, as Lemma 3.2.4 shows that if $DF_{m+1}(S^1 \wedge \dot{\Delta}[r]_+) \to D(C\lambda_m \wedge_S F_0(\dot{\Delta}[r]_+))$ is a π_*-isomorphism then the pushout $DF_{m+1}(S^1 \wedge \Delta[r]_+) \to DP_{m,r}$ is also a π_*-isomorphism. If

$DF_{m+1}(S^1 \wedge \Delta[r]_+) \to D(C\lambda_m \wedge_S F_0(\Delta[r]_+))$ is also a π_*-isomorphism, this implies that $DP_{m,r} \to D(C\lambda_m \wedge_S F_0(\Delta[r]_+))$ is a π_*-isomorphism.

Since F_0K is cofibrant and $C\lambda_m \to F_mS^0$ is a level equivalence, the map $C\lambda_m \wedge_S F_0K \to F_mS^0 \wedge_S F_0K = F_mK$ is a level equivalence, by Lemma 2.1.15. As already noticed, D takes level equivalences to π_*-isomorphisms so we can assume that $C\lambda_m$ is replaced by F_mS^0 in c_K for both values of K.

So we show that $Dc_K\colon\ DF_{m+1}(S^1 \wedge K) \to DF_mK$ is a π_*-isomorphism. Note that $F_{m+1}(S^1 \wedge K) \to F_mK$ is induced by $\hom_I(m+1, n) \to \hom_I(m, n)$ which, in turn, is induced by the inclusion of m in $m+1$. Now consider homotopy applied to the map of diagrams, $\mathcal{D}c_K$. Using the π_*-isomorphisms from Lemma 3.2.5 above, this map is a map of free diagrams, $\hom_I(m+1, -) \otimes \pi^s_{*+m+1}S^1 \wedge K \to \hom_I(m, -) \otimes \pi^s_{*+m}K$. This map induces an isomorphism on the colimits and all of the higher colim^i vanish. Hence, using the spectral sequence for calculating the homotopy of homotopy colimits (see Section 2.2) Dc_K is a π_*-isomorphism. One can also see this by considering the associated map of free diagrams directly. □

4. Topological Hochschild Homology

Let k be a commutative symmetric ring spectrum. Let R be a k-algebra. Define $R^e = R \wedge_k R^{op}$. Let M be a k-symmetric R-bimodule, i.e., an R^e-module. With this set up we have two different algebraic definitions of topological Hochschild homology, one using a derived tensor product definition, the other mimicking the usual Hochschild complex. In Theorem 4.1.10 we see that these definitions construct stably equivalent k-modules. Of course, since the smash product is only stably invariant for cofibrant spectra, the case where R is a cofibrant k-module is the only one of interest.

The idea to define topological Hochschild homology (THH) by mimicking algebra in this way is due to Goodwillie [9]. But because a symmetric monoidal category of spectra was not available until recently, one could not simply implement this idea. Bökstedt was the first one to define THH by modifying this idea to work with certain rings up to homotopy. This original definition of THH concerns the case when $k = S$. We restate the definition of the simplicial spectrum $\mathrm{THH}.(R)$ and its realization, $\mathrm{THH}(R)$, from [1] for a symmetric ring spectrum (see Definition 4.2.6). Theorem 4.2.8 shows that for $k = S$ the new definitions are stably equivalent to the original definition when R is a cofibrant symmetric ring spectrum. As a corollary to this comparison theorem we see that Bökstedt's definition of THH takes stable equivalences of S-algebras to π_*-isomorphisms. Hence it always determines the right homotopy type, even on non-connective and non-convergent ring spectra, that is, without Bökstedt's original hypotheses. As noted above, the other two algebraic definitions give the right homotopy type only on cofibrant symmetric ring spectra. To avoid these unnecessary hypotheses we use model category techniques and the detection functor developed in Section 3.

4.1. TWO DEFINITIONS OF RELATIVE TOPOLOGICAL HOCHSCHILD HOMOLOGY

The first definition corresponds to the derived tensor product notion of algebraic Hochschild homology. The second definition mimics the Hochschild complex from algebraic Hochschild homology. As we see in Theorem 4.1.10, these notions are stably equivalent when M is a cofibrant R^e-module.

DEFINITION 4.1.1. Define $\mathrm{thh}^{\mathrm{k}}(R; M)$ by $M \wedge_{R^e} R$.

Let $\mu\colon R \wedge_k R \to R$ and $\eta\colon k \to R$ be the multiplication and unit maps on R. Let $\phi_r\colon M \wedge_k R \to M$ and $\phi_l\colon R \wedge_k M \to M$ be the right and left R-module structure maps of R acting on M. Let R^s be the smash product over k of s copies of R, i.e., $R \wedge_k \cdots \wedge_k R$. The following definition mimics the Hochschild complex as in [4].

DEFINITION 4.1.2. $\mathrm{tHH}^{\mathrm{k}}_{\cdot}(R; M)$ is the simplicial k-module with s-simplices $M \wedge_k R^s$. The simplicial face and degeneracy maps are given by

$$d_i = \begin{cases} \phi_r \wedge (\mathrm{id}_R)^{s-1} & \text{if } i = 0, \\ (\mathrm{id}_M) \wedge (\mathrm{id}_R)^{i-1} \wedge \mu \wedge (\mathrm{id}_R)^{s-i-1} & \text{if } 1 \leqslant i < s, \\ (\phi_l \wedge (\mathrm{id}_R)^{s-1}) \circ \tau & \text{if } i = s, \end{cases}$$

and $s_i = \mathrm{id}_M \wedge (\mathrm{id}_R)^i \wedge \eta \wedge (\mathrm{id}_R)^{s-1}$.

Each level of this simplicial symmetric spectrum is a bisimplicial set. Since the realization of bisimplicial sets is equivalent to taking the diagonal, we use the diagonal to define the realization of this simplicial symmetric spectrum.

DEFINITION 4.1.3. Define the k-module $\mathrm{tHH}^{\mathrm{k}}(R; M)$ as the diagonal of the bisimplicial set at each level of this simplicial k-module. For the special cases of $k = S$ or $M = R$ we delete them from the notation.

Since the homotopy colimit of a diagram of symmetric spectra is determined by the homotopy colimit of each level, the fact that the homotopy colimit of a bisimplicial set is weakly equivalent to the diagonal simplicial set, see [3, XII 4.3], proves the following proposition.

PROPOSITION 4.1.4. *The map* $\mathrm{hocolim}^{\Delta^{op}}_{Sp^\Sigma} \mathrm{tHH}^{\mathrm{k}}_{\cdot}(R; M) \to \mathrm{tHH}^{\mathrm{k}}(R; M)$ *is a level equivalence.*

Next we show certain homotopy invariance properties of $\mathrm{tHH}^{\mathrm{k}}$. First we show the realization of a map which is a stable equivalence at each simplicial level is a stable equivalence.

LEMMA 4.1.5. *Let* $F, G\colon\ B \to Sp^{\Sigma}$ *be two diagrams of symmetric spectra with a natural transformation* $\eta\colon\ F \to G$ *between them. If* $\eta(b)\colon\ F(b) \to G(b)$ *is a stable equivalence for each object* b *in* B *then* $\mathrm{hocolim}^B F \to \mathrm{hocolim}^B G$ *is a stable equivalence.*

Proof. Consider $D\eta\colon\ DF \to DG$. By Theorem 3.1.2 this is a π_*-isomorphism at each object, so by Proposition 2.2.5 the homotopy colimits are π_*-isomorphic. Since L', F_0 and homotopy colimits commute with homotopy colimits and Ω^n commutes with homotopy colimits up to π_*-isomorphism, $\mathrm{hocolim}^B DF$ is π_*-isomorphic to $D\,\mathrm{hocolim}^B F$. Hence, $D\,\mathrm{hocolim}^B F \to D\,\mathrm{hocolim}^B G$ is a π_*-isomorphism. Thus, by Theorem 3.1.2, $\mathrm{hocolim}^B F \to \mathrm{hocolim}^B G$ is a stable equivalence. □

COROLLARY 4.1.6. *A map between simplicial symmetric spectra which is a stable equivalence on each level induces a stable equivalence on the realizations.*

Proof. This just combines Lemma 4.1.5 and Proposition 4.1.4 or [3, XII, 4.3]. □

PROPOSITION 4.1.7. *Let* $R \to R'$ *be a stable equivalence between* k*-algebras which are cofibrant as* k*-modules,* M *an* R^e*-module,* N *an* $(R')^e$*-module, and* $M \to N$ *a stable equivalence of* R^e*-modules. Then* $\mathrm{tHH}^{\mathrm{k}}(R; M) \to \mathrm{tHH}^{\mathrm{k}}(R'; N)$ *is a stable equivalence. In particular,* $\mathrm{tHH}^{\mathrm{k}}(R) \to \mathrm{tHH}^{\mathrm{k}}(R')$, $\mathrm{tHH}^{\mathrm{k}}(R; M) \to \mathrm{tHH}^{\mathrm{k}}(R; N)$*, and* $\mathrm{tHH}^{\mathrm{k}}(R; N) \to \mathrm{tHH}^{\mathrm{k}}(R'; N)$ *are stable equivalences.*

First note that a cofibrant k-algebra is also cofibrant as a k-module by [11, 5.4.3], so there are many examples of k-algebras which are cofibrant as k-modules.

Proof. Lemma 2.1.15 applied to k shows that $P \wedge_k -$ preserves stable equivalences of k-modules if P is a cofibrant k-module. Hence, $R^s \to R'^s$ is a stable equivalence between cofibrant k-modules. So both $M \wedge_k R^s \to N \wedge_k R^s$ and $N \wedge_k R^s \to N \wedge_k R'^s$ are also stable equivalences. Thus each simplicial level is a stable equivalence. Then Corollary 4.1.6 shows that this map induces a stable equivalence on $\mathrm{tHH}^{\mathrm{k}}$. □

To compare these two definitions of THH we proceed as in [6, IX 2]. Let N be a left R-module, with $\phi_N\colon\ R \wedge_k N \to N$, and M a right R-module, with $\phi_M\colon\ M \wedge_k R \to M$. We define the topological bar construction $B^k_{\cdot}(M, R, N)$ by mimicking algebra.

DEFINITION 4.1.8. The bar construction $B^k_{\cdot}(M, R, N)$ is the simplicial k-module with s-simplices $M \wedge_k R^s \wedge_k N$. The face and degeneracy maps are given by

$$d_i = \begin{cases} \phi_M \wedge (\mathrm{id}_R)^{s-1} \wedge \mathrm{id}_N & \text{if } i = 0,\\ \mathrm{id}_M \wedge (\mathrm{id}_R)^{i-1} \wedge \mu \wedge (\mathrm{id}_R)^{s-i-1} \wedge \mathrm{id}_N & \text{if } 1 \leqslant i < s,\\ \mathrm{id}_M \wedge (\mathrm{id}_R)^{s-1} \wedge \phi_N & \text{if } i = s.\end{cases}$$

Let $B^k(M, R, N)$ be the realization of this simplicial k-module.

Let $c.(X)$ be the constant simplicial object with X in each simplicial degree. Using the identification $M \wedge_R R \cong M$, the map $\eta\colon\ k \to R$ induces a simplicial k-module map $B^k_\cdot(M, R, N) \to c.(M \wedge_R N)$.

LEMMA 4.1.9. *For M a cofibrant R-module, the simplicial map of k-modules, $B^k_\cdot(M, R, N) \to c.(M \wedge_R N)$, induces a stable equivalence of $B^k(M, R, N) \to M \wedge_R N$.*

Proof. Note that $B^k_\cdot(M, R, N) \cong c.M \wedge_R B^k\cdot(R, R, N)$. Since realization commutes with smash products, $B^k(M, R, N) \cong M \wedge_R B^k(R, R, N)$. So, using Lemma 2.1.15, it is enough to show that $B^k(R, R, N) \to N$ is a stable equivalence. The map $N \cong k \wedge_k N \to R \wedge_k N$ provides a simplicial retraction for $B^k_\cdot(R, R, N)$. Hence the spectral sequence for computing the classical stable homotopy groups of the homotopy colimit of this simplicial k-module collapses. So the map $B^k_\cdot(R, R, N) \to c.N$ induces a π_*-isomorphism on the realizations. □

Using the bar construction we show the two definitions of THH are stably equivalent when M is a cofibrant R^e-module.

THEOREM 4.1.10. *There is a natural map of k-modules* $\mathrm{tHH}^{\mathrm{k}}(R; M) \to \mathrm{thh}^{\mathrm{k}}(R; M)$ *which is a stable equivalence for M a cofibrant R^e-module.*

Proof. We show that $\mathrm{tHH}^{\mathrm{k}}(R; M)$ is naturally isomorphic to $M \wedge_{R^e} B^k(R, R, R)$ below. Then the map $\mathrm{tHH}^{\mathrm{k}}(R; M) \to \mathrm{thh}^{\mathrm{k}}(R; M)$ is given by $M \wedge_{R^e} \phi$ for ϕ: $B^k(R, R, R) \to R$. R is always a cofibrant R-module, hence ϕ is a stable equivalence by Lemma 4.1.9. Then Lemma 2.1.15 shows that $M \wedge_{R^e} \phi$ is a stable equivalence since M is a cofibrant R^e-module.

We now show $\mathrm{tHH}^{\mathrm{k}}.(R; M)$ is naturally isomorphic to $c.(M) \wedge_{R^e} B^k_\cdot(R, R, R)$. On each simplicial level there are natural isomorphisms

$$\begin{aligned} M \wedge_k R^s &\cong M \wedge_{R^e} (R^e \wedge_k R^s) \cong M \wedge_{R^e} (R \wedge_k R^s \wedge_k R) \\ &= M \wedge_{R^e} B^k_s(R, R, R). \end{aligned}$$

These isomorphisms commute with the simplicial structure. Hence the simplicial k-modules are naturally isomorphic, so their realizations are also naturally isomorphic. □

4.2. BÖKSTEDT'S DEFINITION OF TOPOLOGICAL HOCHSCHILD HOMOLOGY

We now define the simplicial spectrum $\mathrm{THH}.(R; M)$ and its realization $\mathrm{THH}(R; M)$ following Bökstedt's original definitions. Each of the levels of the simplicial spectrum THH. can be defined for a general symmetric spectrum X. A ring structure is only necessary for defining the simplicial structure. In fact, level k of THH. can be thought of as a functor generalizing D which gives the correct π_*-isomorphism type for the smash product of $k+1$ symmetric spectra. We start by considering each of these levels as a functor of several variables. See Section 2.2 for facts about homotopy colimits.

Let $\mathbf{X}$ denote a sequence of $j+1$ spectra, $X^0, \ldots, X^j$. Define a functor $\mathcal{D}^j\mathbf{X}$ from I^{j+1} to Sp^Σ which at $\mathbf{n} = (n_0, \ldots, n_j)$ takes the value,

$$\mathcal{D}^j\mathbf{X}(\mathbf{n}) = \Omega^n L' F_0(X^0_{n_0} \wedge \ldots \wedge X^j_{n_j}),$$

where L' is a level fibrant replacement functor and $n = \Sigma n_i$, the sum of the n_i. Note that $\mathcal{D}^0(X)$ is $\mathcal{D}_X$, the functor defined at the beginning of Section 3. To see that $\mathcal{D}^j\mathbf{X}$ is defined over I^{j+1} one uses maps similar to those described for $\mathcal{D}_X$.

DEFINITION 4.2.1. Let $X^0, \ldots, X^j$ be symmetric spectra. Define

$$\mathrm{T}_j\,\mathbf{X} = \mathrm{hocolim}^{I^{j+1}}\,\mathcal{D}^j\mathbf{X}.$$

We now define a natural transformation $\phi_j\mathbf{X}\colon\ \mathrm{T}_j\,\mathbf{X} \to D(X^0 \wedge_S \ldots \wedge_S X^j)$. Let $\mu\colon\ I^{j+1} \to I$ be the functor induced by concatenation. Then there is a natural transformation from $\mathcal{D}^j\mathbf{X}$ to $\mu^*\mathcal{D}^0(X^0 \wedge_S \ldots \wedge_S X^j)$. It is induced by the map from $X^0_{n_0} \wedge \cdots \wedge X^j_{n_j}$ to the nth level of $X^0 \wedge_S \ldots \wedge_S X^j$. This map is $\Sigma_{n_0} \times \cdots \times \Sigma n_j$ equivariant, which is exactly what is necessary over I^{j+1}. Hence, on homotopy colimits there is a natural map $\mathrm{hocolim}^{I^{j+1}}\,\mathcal{D}^j\mathbf{X} \to \mathrm{hocolim}^{I^{j+1}}\,\mu^*\mathcal{D}^0(X^0 \wedge_S \ldots \wedge_S X^j)$.

DEFINITION 4.2.2. There is a natural transformation $\phi_j\mathbf{X}\colon\ \mathrm{T}_j\,\mathbf{X} \to D(X^0 \wedge_S \ldots \wedge_S X^j)$. It is given by the composition

$$\mathrm{hocolim}^{I^{j+1}}\,\mathcal{D}^j\mathbf{X} \to \mathrm{hocolim}^{I^{j+1}}\,\mu^*\mathcal{D}^0(X^0 \wedge_S \ldots \wedge_S X^j)$$
$$\to \mathrm{hocolim}^{I}\,\mathcal{D}^0(X^0 \wedge_S \ldots \wedge_S X^j).$$

PROPOSITION 4.2.3. *For any cofibrant symmetric spectra, $X^0, \ldots, X^j$, the map $\phi_j\mathbf{X}$ is a π_*-isomorphism.*

This proposition is proved in Section 4.3. It is used in proving the comparison theorem between Bökstedt's definition of THH and our previous definition of tHH. As a corollary of this proposition, T_j gives the correct π_*-isomorphism type for the derived smash product of $j+1$ symmetric spectra. Recall that the smash product is only homotopy invariant on cofibrant spectra, so the derived smash product is the smash product of the cofibrant replacements. In the stable model category of symmetric spectra, consider a cofibrant replacement functor, C, analogous to the fibrant replacement functor L.

COROLLARY 4.2.4. *$\pi_*\mathrm{T}_j\,\mathbf{X}$ is isomorphic to $\pi_*L(CX^0 \wedge_S \ldots \wedge_S CX^j)$, the derived homotopy of the derived smash product of $X^0, \ldots, X^j$.*

Proof. Since C is a cofibrant replacement functor, $CX \to X$ is a level equivalence. Hence $\mathrm{T}_j(CX^0, \ldots, CX^j) \to \mathrm{T}_j(X^0, \ldots, X^j)$ is a level equivalence by

Proposition 2.2.2, as the map is a level equivalence at each object in the diagram defining T_j. So this corollary follows from Proposition 4.2.3 since $\pi_* D(CX^0 \wedge_S \ldots \wedge_S CX^j)$ is isomorphic to $\pi_* L(CX^0 \wedge_S \ldots \wedge_S CX^j)$ by Theorem 3.1.6. ☐

We now define THH following Bökstedt's definition in [1].

DEFINITION 4.2.5. Let R be a symmetric ring spectrum with M an R^e-module. Define $\mathrm{THH}_j(R; M) = \mathrm{T}_j(M, R, \ldots, R)$.

The functors $\mathrm{THH}_j(R; M)$ fit together to form a simplicial symmetric spectrum $\mathrm{THH}.(R; M)$. Although the definition of $\mathrm{THH}_j(R; M)$ does not use the ring structure of R or the module structure of M, the simplicial structure of $\mathrm{THH}.(R; M)$ does use both the multiplication and unit maps. The ith face map uses the functor $\delta_i\colon\ I^{j+1} \to I^j$ defined by concatenation of the sets in factors i and $i + 1$. The last face map uses the cyclic permutation of I^{j+1} followed by concatenation of the first two factors. For ease of notation let $\mathcal{D}^j(R; M) = \mathcal{D}^j(M, R, \ldots, R)$. The multiplication of R and M defines a natural transformation of functors from $\mathcal{D}^j(R; M)$ to $\delta_i^* \mathcal{D}^{j-1}(R; M)$. So d_i is the composition

$$\begin{aligned} d_i\colon\ & \mathrm{hocolim}^{I^{j+1}}\, \mathcal{D}^j(R; M) \to \mathrm{hocolim}^{I^{j+1}}\, \delta_i^* \mathcal{D}^{j-1}(R; M) \\ & \to \mathrm{hocolim}^{I^j}\, \mathcal{D}^{j-1}(R; M). \end{aligned}$$

The degeneracy maps are similar.

DEFINITION 4.2.6. Define $\mathrm{THH}(R; M)$ as the diagonal of the bisimplicial set at each level of the simplicial symmetric spectrum $\mathrm{THH}.(R; M)$.

One can check that each level in this spectrum agrees with the definition in [1] when $M = R$.

As in Proposition 4.1.4 we have the following equivalence.

PROPOSITION 4.2.7. *The map* $\mathrm{hocolim}^{\Delta^{op}}_{Sp^\Sigma} \mathrm{THH}.(R; M) \to \mathrm{THH}(R; M)$ *is a level equivalence.*

The next theorem shows that the definition of THH which mimics the Hochschild complex is stably equivalent to the original definition of THH.

THEOREM 4.2.8. *Let R be a cofibrant ring spectrum. Then there is a natural zig-zag of stable equivalences between* $\mathrm{tHH}(R; M)$ *and* $\mathrm{THH}(R; M)$.

Proof. The zig-zag of functors between tHH and THH is induced by a zig-zag of maps between the simplicial complexes defining tHH and THH. First one applies the zig-zag of functors $1 \xrightarrow{\psi} L \to ML \leftarrow DL \xleftarrow{D\psi} D$ to each simplicial level of the Hochschild complex defining tHH. Here, L is the fibrant replacement functor, M, D, and the natural transformations are defined in Section 3, see 3.1.1, 3.2.1, and the proof of 3.1.9 part 3. Then there is a natural map $\phi_j\colon\ \mathrm{THH}_j(R; M) \to$

$D(M \wedge_S R^j)$. To see that the ϕ_j maps commute with the simplicial maps, one needs to note that the multiplication maps commute with the first map in the composite defining ϕ_j. This follows since the map $R_n \wedge R_m \to R_{n+m}$ is the map on the appropriate wedge summand of the map $R \otimes R \to R$ which induces the map $R \wedge_S R \to R$. The maps involving M are similar. Putting these simplicial levels together one gets a zig-zag of natural transformations from $\mathrm{tHH}_\cdot(-)$ to $\mathrm{THH}_\cdot(-)$.

The zig-zag of functors between 1 and D was investigated in Section 3. Each functor induces a stable equivalence on each simplicial level, by Definition 2.1.3, Theorem 3.1.2, and Proposition 3.1.9. Corollary 4.1.6 shows that they induce stable equivalences on the realizations.

So the only part left is $\phi_\cdot\colon\ \mathrm{THH}_\cdot(R; M) \to D(\mathrm{tHH}_\cdot(R; M))$. Let $CM \to M$ be a cofibrant replacement of M as an R^e-module. Then by Proposition 4.1.7, $\mathrm{tHH}_j(R; CM) \to \mathrm{tHH}_j(R; M)$ is a stable equivalence. Similarly, $\mathrm{THH}_j(R; CM) \to \mathrm{THH}_j(R; M)$ is a stable equivalence since $CM \to M$ is a level equivalence and hence induces a level equivalence on the homotopy colimits used to define THH_j. So we can assume M is cofibrant as an R^e-module.

Since R is cofibrant as an S-algebra, it is also cofibrant as an S-module. Since M is cofibrant as an R^e-module and R^e is cofibrant, M is also cofibrant as an S-module. Proposition 4.2.3 shows that if R and M are any cofibrant S-modules then $\mathrm{THH}_j(R; M) \to D(M \wedge_S R^j)$ is a π_*-isomorphism. By Proposition 2.2.5 the map of realizations is a π_*-isomorphism. Hence, assuming Proposition 4.2.3, this finishes the proof of Theorem 4.2.8. □

Using this comparison we show Bökstedt's original definition of THH takes stable equivalences of ring spectra to π_*-isomorphisms. This is a stronger result than for tHH because no cofibrancy condition is needed here and the map is a π_*-isomorphism, not just a stable equivalence.

COROLLARY 4.2.9. *Let $R \to R'$ be a stable equivalence of ring spectra, M an R^e-module, N an $(R')^e$-module, and $M \to N$ a stable equivalence of R^e-modules. Then* $\mathrm{THH}(R; M) \to \mathrm{THH}(R'; N)$ *is a π_*-isomorphism.*

Remark 4.2.10. This corollary could also be proved without using these comparison results. Each THH_j takes stable equivalences to π_*-isomorphisms by arguments similar to those for $\mathrm{THH}_0 = D$ in Section 3. By Proposition 2.2.5 the realization, THH, also takes stable equivalences to π_*-isomorphisms.

Proof. In the category of symmetric ring spectra, define a functorial cofibrant replacement functor, C. Applying this functor we have the following square.

$$\begin{array}{ccc} CR & \longrightarrow & CR' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

Each of the vertical maps is a level trivial fibration and hence a level equivalence. The bottom map is a stable equivalence by assumption. Hence the top map is also a stable equivalence. To show THH applied to the bottom map is a π_*-isomorphism we show THH applied to the other three maps in this square are π_*-isomorphisms.

We also consider cofibrant replacements of the modules in question. M is a $(CR)^e$-module and N is a $(CR')^e$-module. Since CR is a cofibrant S-algebra it is a cofibrant S-module. Thus, $(CR)^e$ is also cofibrant as an S-module by the monoidal structure of the stable model category [11, 5.3.8]. Hence the cofibrations in the category of $(CR)^e$-modules are also underlying cofibrations. So let $CM \to M$ be the cofibrant replacement of M in the category of $(CR)^e$-modules. Similarly, let $CN \to N$ be the cofibrant replacement of N as a $(CR')^e$-module. Then both CM and CN are cofibrant as S-modules. Also, by the lifting property in the model category of $(CR)^e$-modules, we have a map $CM \to CN$ because $CN \to N$ is a level trivial fibration. This map $CM \to CN$ is a stable equivalence by the two out of three property.

The level equivalences $CR \to R$ and $CM \to M$ induce a level equivalence on each object of the diagram defining THH_j. So by applying Proposition 2.2.2 and Lemma 4.2.7 this shows that $\mathrm{THH}(CR; CM) \to \mathrm{THH}(R; M)$ is a level equivalence. Similarly $\mathrm{THH}(CR'; CN) \to \mathrm{THH}(R'; N)$ is a level equivalence.

For the top map, first consider applying tHH. Proposition 4.1.7 implies that $\mathrm{tHH}(CR; CM) \to \mathrm{tHH}(CR'; CN)$ is a stable equivalence. Hence by Theorem 3.1.2, $D\,\mathrm{tHH}(CR; CM) \to D\,\mathrm{tHH}(CR'; CN)$ is a π_*-isomorphism. But, in the proof of Theorem 4.2.8, we showed that $\mathrm{THH} \to D\,\mathrm{tHH}$ induces a π_*-isomorphism if the ring and module are cofibrant as S-modules. So $\mathrm{THH}(CR; CM) \to \mathrm{THH}(CR'; CN)$ is a π_*-isomorphism. Stringing these equivalences together finishes the proof of this corollary. As Proposition 4.2.3 applies to each level, we have actually shown that each $\mathrm{THH}_j(R; M) \to \mathrm{THH}_j(R'; N)$ is also a π_*-isomorphism. □

4.3. PROOF OF PROPOSITION 4.2.3

To prove Proposition 4.2.3 we follow an outline similar to the proof that D takes stable trivial cofibrations to π_*-isomorphisms, see Section 3.2. We show that ϕ_j is a π_*-isomorphism when it is evaluated only on free symmetric spectra, i.e., some $F_n K$. Then we prove an induction step lemma which deals with pushouts over generating stable trivial cofibrations. Using these lemmas we show ϕ_j is a π_*-isomorphism on any collection of cofibrant spectra.

LEMMA 4.3.1. *$\phi_j(F_{n_0}K_0, \ldots, F_{n_j}K_j)$ is a π_*-isomorphism.*

LEMMA 4.3.2. *Let $A \to B$ be a stable cofibration and $X^0, \ldots, X^j$ be cofibrant S-modules. Consider the following pushout square.*

$$\begin{array}{ccc} A & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

Assume that $\mathrm{THH}_{j+1}(X^0, \ldots, Z, \ldots, X^j) \to D(X^0 \wedge_S \ldots \wedge_S Z \wedge_S \ldots \wedge_S X^j)$ *is a* π_**-isomorphism for* $Z = A, B,$ *or* W *where* Z *is inserted between the* i*th and* $(i+1)$*th spots. Then* $\mathrm{THH}_{j+1}(X^0, \ldots, Y, \ldots, X^j) \to D(X^0 \wedge_S \ldots \wedge_S Y \wedge_S \ldots \wedge_S X^j)$ *is a* π_**-isomorphism.*

Using these two lemmas we can now prove Proposition 4.2.3.

Proof of Proposition 4.2.3. We prove this by induction on i with the induction assumption that ϕ_j is a π_*-isomorphism when $j - i$ variables are free spectra and the other variables are cofibrant. Lemma 4.3.1 verifies this for $i = 0$. For the induction step, in one variable we build up a cofibrant spectrum from the initial spectrum by retracts, colimits, and pushouts over generating cofibrations. Since retracts of π_*-isomorphisms are π_*-isomorphisms and ϕ_j of a retract is a retract we only need to consider colimits and pushouts.

As F_0, smash products, L', Ω^n, and homotopy colimits commute with filtered colimits, T_j of a colimit in one of the variables is a colimit. This is also true of D. Since a filtered colimit of π_*-isomorphisms is a π_*-isomorphism, ϕ_j of a colimit in one variable is a π_*-isomorphism if it is a π_*-isomorphism at each spot in the sequence. Hence we are only left with pushouts.

Since ϕ_j is a level equivalence between trivial spectra if one of the variables is the initial spectrum, $*$, proceed by induction to verify the pushout property. By induction the two corners in the pushout corresponding to the generating cofibration are π_*-isomorphisms. This is because generating cofibrations are of the form $F_n K \to F_n L$, so these two corners have $j - i + 1$ free spectra and hence, fall into the case covered by the previous induction step. The third corner is assumed to be a π_*-isomorphism by induction. Hence ϕ_j is a π_*-isomorphism on the pushout corner by Lemma 4.3.2. □

Proof of Lemma 4.3.1. We first establish the stable homotopy type of $\mathrm{T}_j(F_{n_0}K_0, \ldots, F_{n_j}K_j)$. There is a free diagram functor $\mathcal{F}_{(n_0,\ldots,n_j)}X\colon\ I^{j+1} \to Sp^\Sigma$ defined by

$$\mathcal{F}_{(n_0,\ldots,n_j)}X(m_0, \ldots m_j) = \mathrm{hom}_{I^{j+1}}((n_0, \ldots, n_j), (m_0, \ldots m_j))_+ \wedge X.$$

Then $\mathcal{F}_{(n_0,\ldots,n_j)}(-)$ is left adjoint to the functor from I-diagrams over Sp^Σ to Sp^Σ which evaluates the diagram at $(n_0, \ldots, n_j) \in I^{j+1}$. There is a map of diagrams

$$\mathcal{F}_{(n_0,\ldots,n_j)}(\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j)) \to \mathcal{D}^j(F_{n_0}K_0, \ldots, F_{n_j}K_j)$$

where $n = \Sigma n_i$. Each spot in this diagram is a π_*-isomorphism. This is similar to the proof of Lemma 3.2.5, on each level the map is an equivalence in the stable range by the Blakers–Massey and the Freudenthal suspension theorems. Hence the map on homotopy colimits is also a π_*-isomorphism,

$$\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j)) \to \mathrm{T}_j(F_{n_0}K_0, \ldots, F_{n_j}K_j).$$

By Lemma 3.2.5,

$$\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j)) \to D(F_n(K_0 \wedge \ldots \wedge K_j))$$

is also a π_*-isomorphism. To see that ϕ_j induces a π_*-isomorphism, note that on the free diagrams there are similar maps

$$\begin{aligned}&\mathrm{hocolim}^{I^{j+1}} \mathcal{F}_{(n_0,\ldots,n_j)}(\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j))\\ &\quad\to \mathrm{hocolim}^{I^{j+1}} \mu^* \mathcal{F}_n(\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j))\\ &\quad\to \mathrm{hocolim}^{I} \mathcal{F}_n(\Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j))\end{aligned}$$

which induce level equivalences on the homotopy colimits. □

To prove Lemma 4.3.2 we first show T_j of a homotopy pushout in one variable is a homotopy pushout.

LEMMA 4.3.3. *Let $X^0, \ldots, X^j$ be cofibrant spectra. If*

$$\begin{array}{ccc} A & \longrightarrow & W \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is a pushout square with $A \to B$ a cofibration, then

$$\begin{array}{ccc} \mathrm{T}_{j+1}(X^0, \ldots, A, \ldots X^j) & \longrightarrow & \mathrm{T}_{j+1}(X^0, \ldots, W, \ldots X^j) \\ \downarrow & & \downarrow \\ \mathrm{T}_{j+1}(X^0, \ldots, B, \ldots X^j) & \longrightarrow & \mathrm{T}_{j+1}(X^0, \ldots, Y, \ldots X^j) \end{array}$$

is a homotopy pushout square. That is, if P is the homotopy pushout of the second square then $P \to \mathrm{T}_{j+1}(X^0, \ldots, Y, \ldots X^j)$ is a stable equivalence. In fact, $P \to \mathrm{T}_{j+1}(X^0, \ldots, Y, \ldots X^j)$ is a π_*-isomorphism.

Proof. This proof is similar to the proof of Lemma 3.2.3. As with Lemma 3.2.3, it is enough to consider each object in I^{j+1} since homotopy colimits commute.

The following square is a pushout square with the left map a cofibration:

$$\begin{array}{ccc} X^0_{n_0} \wedge \ldots \wedge A_{n_i} \wedge \ldots X^j_{n_j} & \longrightarrow & X^0_{n_0} \wedge \ldots \wedge W_{n_i} \wedge \ldots X^j_{n_j} \\ \downarrow & & \downarrow \\ X^0_{n_0} \wedge \ldots \wedge B_{n_i} \wedge \ldots X^j_{n_j} & \longrightarrow & X^0_{n_0} \wedge \ldots \wedge Y_{n_i} \wedge \ldots X^j_{n_j} \end{array}$$

The first step in constructing T_j is just applying F_0 to this square. F_0 preserves cofibrations and pushouts, hence F_0 applied to this square is a homotopy pushout. L' preserves homotopy pushout squares up to level equivalence and $\Omega^{\Sigma n_i}$ preserves homotopy pushout squares up to π_*-isomorphism. Hence the map from the homotopy pushout to the bottom right corner is a π_*-isomorphism. Since the homotopy colimit of π_*-isomorphisms is a π_*-isomorphism, this finishes the proof. □

Proof of Lemma 4.3.2. Both T_j and D take homotopy pushouts in one variable to homotopy pushouts where the map from the pushout to the bottom right corner is a π_*-isomorphism by Lemmas 4.3.3 and 3.2.3. Hence, this lemma follows from the fact that homotopy colimits preserve π_*-isomorphisms, Lemma 2.2.5. □

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