Exercises on derived categories, resolutions, and Brown representability

Henning Krause

The numbering of the following exercises refers to the article "Derived categories, resolutions, and Brown representability" in this volume.

(1.2.1) Let $\mathcal{A}$ be an abelian category. Show that $\mathbf{K}(\mathcal{A})$ and $\mathbf{D}(\mathcal{A})$ are additive categories and that the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is additive.

(1.4.1) Let $\mathcal{A}$ be an abelian category and denote by T the class of all quasi-isomorphisms in $\mathbf{C}(\mathcal{A})$. Show that two maps $\phi, \psi \colon X \to Y$ in $\mathbf{C}(\mathcal{A})$ are identified by the canonical functor $\mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})[T^{-1}]$ if $\phi - \psi$ is null-homotopic.

(1.5.1) Let $\mathcal{A}$ be the module category of a ring Λ. Show that $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(\Lambda, X) \cong H^0 X$ for every complex X of Λ-modules.

(1.5.2) Let $\mathcal{A}$ be an abelian category. Show that the canonical functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ identifies $\mathcal{A}$ with the full subcategory of complexes X in $\mathbf{D}(\mathcal{A})$ such that $H^n X = 0$ for all $n \neq 0$.

(1.6.1) Let $\mathcal{A}$ be the category of vector spaces over a field k. Describe all objects and morphisms in $\mathbf{D}(\mathcal{A})$.

(1.6.2) Let $\mathcal{A}$ be the category of finitely generated abelian groups and $\mathcal{P}$ be the category of finitely generated free abelian groups. Describe all objects and morphisms in $\mathbf{D}^b(\mathcal{A})$. Show that the canonical functor $\mathbf{K}^b(\mathcal{P}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence.

(1.6.3) Let k be a field and consider the following finite dimensional algebras.

$$\Lambda_1 = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} k & k & 0 \\ 0 & k & 0 \\ 0 & k & k \end{bmatrix} \quad \Lambda_3 = \Lambda_1/I, \quad I = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Describe in each case the category $\mathcal{A}_i$ of finite dimensional Λ_i-modules and its derived category $\mathbf{D}^b(\mathcal{A}_i)$. Here are some hints.

1. $\mathcal{A}_1$ and $\mathcal{A}_2$ are hereditary categories, but $\mathcal{A}_3$ is not.
2. Each object in $\mathcal{A}_i$ or $\mathbf{D}^b(\mathcal{A}_i)$ decomposes essentialy uniquely into a finite number of indecomposable objects.
3. The indecomposable projective Λ_i-modules are $E_{jj}\Lambda_i$, $j = 1, 2, 3$.
4. Λ_1 and Λ_2 have each 6 pairwise non-isomorphic indecomposable modules, and Λ_3 has 5.

(5) $\mathrm{Ext}^n_{\Lambda_i}(X,Y)$ has k-dimension at most 1 for all indecomposable Λ_i-modules X,Y and $n \geq 0$.

The *Auslander-Reiten quiver* provides a convenient method to display the categories $\mathcal{A}_i$ and $\mathbf{D}^b(\mathcal{A}_i)$, because the morphism spaces between indecomposable objects are at most one-dimensional. This quiver (=oriented graph) is defined as follows. The vertices correspond to the indecomposable objects. Put an arrow $X \to Y$ between two indecomposable objects if there is an irreducible map $\phi\colon X \to Y$ (where ϕ is *irreducible* if ϕ is not invertible and any factorization $\phi = \phi'' \circ \phi'$ implies that ϕ' is a split monomorphism or ϕ'' is a split epimorphism).

(1.7.1) Let $\mathcal{A}$ be an abelian category. Show that the canonical functor $\mathbf{D}^b(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is fully faithful.

(1.7.2) Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{I}$ the full subcategory of injective objects. Suppose that $\mathcal{A}$ has enough injective objects. Then the canonical functor $\mathbf{K}^+(\mathcal{I}) \to \mathbf{D}^+(\mathcal{A})$ is an equivalence.

(1.7.3) Let $\mathcal{A}$ be the category of finite dimensional modules over $\Lambda = k[T]/(T^2)$, where k is a field. Describe the derived category $\mathbf{D}^b(\mathcal{A})$. (Hint: Fix an injective resulution I of the unique simple module $k[T]/(T)$ (with $I^n = \Lambda$ or $I^n = 0$ for all n) and build every object in $\mathbf{D}^b(\mathcal{A})$ from I.)

(2.1.1) Let $\mathcal{T}$ be a triangulated category. Show that the coproduct of two exact triangles is an exact triangle. Generalize this as follows. Let $X_i \to Y_i \to Z_i \to \Sigma X_i$ be a family of exact triangles such that the coproducts $\coprod_i X_i$, $\coprod_i Y_i$, and $\coprod_i X_i$ exist in $\mathcal{T}$. Show that

$$\coprod_i X_i \longrightarrow \coprod_i Y_i \longrightarrow \coprod_i Z_i \longrightarrow \Sigma(\coprod_i X_i)$$

is an exact triangle in $\mathcal{T}$.

(2.1.2) Let $\mathcal{T}$ be a triangulated category. Show that the opposite category $\mathcal{T}^{\mathrm{op}}$ is also triangulated.

(2.3.1) Show that every monomorphism $\phi\colon X \to Y$ in a triangulated category has a left inverse ϕ' such that $\phi' \circ \phi = \mathrm{id}_X$.

(2.4.1) Give an example of an exact triangle Δ and two endomorphisms $(\phi_1, \phi_2, \phi_3')$ and $(\phi_1, \phi_2, \phi_3'')$ of Δ such that $\phi_3' \neq \phi_3''$.

(2.5.1) Let $\mathcal{A}$ be an additive category. Check the axioms (TR1) – (TR4) for $\mathbf{K}(\mathcal{A})$.

(3.1.1) Let $\mathcal{A}$ be an abelian category. Show that a map in $\mathbf{K}(\mathcal{A})$ is a quasi-isomorphism if and only if the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ sends the map to an isomorphism in $\mathbf{D}(\mathcal{A})$.

(3.2.1) Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories. Show that a right adjoint of F is an exact functor.

(3.2.2) Let $\mathcal{A}$ be an abelian category. Find a criterion such that $\mathbf{D}(\mathcal{A})$ is an abelian category.

(3.3.1) Let Λ be a noetherian ring and $\mathcal{A}$ be the category of Λ-modules. A complex X in $\mathcal{A}$ has *finite cohomology* if $H^n X$ is finitely generated for all n and vanishes for almost all $n \in \mathbb{Z}$. Show that the complexes with finite cohomology form a thick subcategory of $\mathbf{D}(\mathcal{A})$.

(3.3.2) Let $\mathcal{A}$ be the category of finite dimensional modules over $k[T]/(T^n)$. Describe the thick subcategory of all acyclic complexes in $\mathbf{K}(\mathcal{A})$ which have projective components. Draw the Auslander-Reiten quiver of this category. (Hint: Note that projective and injective modules over $k[T]/(T^n)$ coincide. Each acyclic complex X of injectives is essentially determined by the module $Z^0 X$.)

(3.5.1) Let Λ be a ring and $e = e^2 \in \Lambda$ be an idempotent. Let $\Gamma = e\Lambda e \cong \operatorname{End}_\Lambda(e\Lambda)$. Then $\operatorname{Hom}_\Lambda(e\Lambda, -)$ induces an exact functor $\operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ which extends to an exact functor $F\colon \mathbf{D}(\operatorname{Mod}\Lambda) \to \mathbf{D}(\operatorname{Mod}\Gamma)$. Show that F induces an equivalence

$$\mathbf{D}(\operatorname{Mod}\Lambda)/\operatorname{Ker} F \to \mathbf{D}(\operatorname{Mod}\Gamma).$$

(4.1.1) Let $\mathcal{A}$ be an additive category. Give a presentation of the cokernel of a map between two coherent functors in $\widehat{\mathcal{A}}$.

(4.1.2) Let $\mathcal{A}$ be an additive category. Show that for every family of functors F_i in $\widehat{\mathcal{A}}$ having a presentation

$$\mathcal{A}(-, X_i) \xrightarrow{(-,\phi_i)} \mathcal{A}(-, Y_i) \longrightarrow F_i \longrightarrow 0,$$

the coproduct $F = \coprod_i F_i$ in $\widehat{\mathcal{A}}$ has a presentation

$$\mathcal{A}(-, \coprod_i X_i) \xrightarrow{(-,\amalg\phi_i)} \mathcal{A}(-, \coprod_i Y_i) \longrightarrow F \longrightarrow 0.$$

(4.1.3) Let Λ be a ring and $\mathcal{A}$ be the category of free Λ-modules. Show that $\widehat{\mathcal{A}}$ is equivalent to the category of Λ-modules.

(4.2.1) Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories. Show that the induced functor $\widehat{\mathcal{T}} \to \widehat{\mathcal{U}}$ is exact.

(4.5.1) Let $\mathcal{A}$ be the category of Λ-modules over a ring Λ. Show that Λ is a perfect generator for $\mathbf{D}(\mathcal{A})$.

(4.5.2) Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Show that one can replace in the definition of a perfect generator the condition

(PG1) There is no proper full triangulated subcategory of $\mathcal{T}$ which contains S and is closed under taking coproducts.

by the following condition

(PG1') Let X be in $\mathcal{T}$ and suppose $\operatorname{Hom}_\mathcal{T}(\Sigma^n S, X) = 0$ for all $n \in \mathbb{Z}$. Then $X = 0$.

(5.1.1) Let $\mathcal{A}$ be an abelian category and I be the injective resolution of an object A. Show that the canonical map $A \to I$ induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(I, X) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, X)$$

for every complex X with injective components.

(5.1.2) Let $\mathcal{A}$ be an abelian category and suppose $\mathcal{A}$ has arbitrary products. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ preserves products if and only if products in $\mathcal{A}$ are exact.

(5.1.3) Let $\mathcal{A}$ be an abelian category with a projective generator. Show that products in $\mathcal{A}$ are exact.

(5.1.4) Let $\mathcal{A}$ be an abelian category with arbitrary products, and denote by $\operatorname{Inj}\mathcal{A}$ the full subcategory of injective objects. Show that

$$\mathbf{K}^+(\operatorname{Inj}\mathcal{A}) \subseteq \mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \subseteq \mathbf{K}(\operatorname{Inj}\mathcal{A}).$$

(Hint: Write every complex in $\mathbf{K}^+(\operatorname{Inj}\mathcal{A})$ as a homotopy limit of truncations from $\mathbf{K}^b(\operatorname{Inj}\mathcal{A})$.)

(5.1.5) Let $\mathcal{A}$ be an abelian category with exact products and an injective cogenerator. Denote by $\operatorname{Inj}\mathcal{A}$ the full subcategory of injective objects. Suppose every object in $\mathcal{A}$ has finite injective dimension. Show that $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) = \mathbf{K}(\operatorname{Inj}\mathcal{A})$. In particular, $\mathbf{K}(\operatorname{Inj}\mathcal{A})$ and $\mathbf{D}(\mathcal{A})$ are equivalent. (Hint: An acyclic complex of injectives is null-homotopic.)

(5.1.6) If a ring Λ has finite global dimension, then $\mathbf{K}(\operatorname{Inj}\Lambda)$ and $\mathbf{K}(\operatorname{Proj}\Lambda)$ are equivalent.

(5.3.1) Consider the setup from (1.6.3). Define Λ_1-modules

$$B = E_{11}\Lambda_1 \amalg E_{22}\Lambda_1 \amalg (E_{22}\Lambda_1/E_{23}\Lambda_1) \quad \text{and} \quad C = (E_{11}\Lambda_1/E_{12}\Lambda_1) \amalg E_{11}\Lambda_1 \amalg E_{33}\Lambda_1.$$

Show that $\Lambda_2 \cong \operatorname{End}_{\Lambda_1}(B)$ and $\Lambda_3 \cong \operatorname{End}_{\Lambda_1}(C)$. Viewing these isomorphisms as identifications, we have bimodules ${}_{\Lambda_2}B_{\Lambda_1}$ and ${}_{\Lambda_3}C_{\Lambda_1}$ which induce equivalences

$$\mathbf{R}\operatorname{Hom}_{\Lambda_1}(B, -)\colon \mathbf{D}^b(\mathcal{A}_1) \to \mathbf{D}^b(\mathcal{A}_2) \quad \text{and} \quad \mathbf{R}\operatorname{Hom}_{\Lambda_1}(C, -)\colon \mathbf{D}^b(\mathcal{A}_1) \to \mathbf{D}^b(\mathcal{A}_3).$$

(The Λ_1-modules B and C are examples of so-called tilting modules.)

(6.1.1) Let k be a field and consider again the algebra

$$\Lambda = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix}.$$

Denote by $S = S_1 \amalg S_2 \amalg S_3$ the coproduct of the three simple Λ-modules. Let $P = \mathbf{p}S$ be a projective resolution of S. Compute $A = \mathcal{E}nd_\Lambda(P)$ and show that $H^n A \cong \operatorname{Ext}^n_\Lambda(S, S)$ for all n. Show that $X \mapsto \mathcal{H}om_\Lambda(P, X)$ induces a functor $\mathbf{K}(\operatorname{Proj}\Lambda) \to \mathbf{D}_{\mathrm{dg}}(A)$ which is an equivalence.

(6.2.1) View a k-algebra A as a category $\mathcal{A}$ with a single object $*$ and $\mathcal{A}(*, *) = A$. Establish an equivalence between the category of right A-modules and the category of k-linear functors $\mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$.

(6.5.1) Let $\mathcal{A}$ be the module category of a noetherian ring, and let A in $\mathcal{A}$ be finitely generated. Show that A is a compact object in $\mathcal{A}$. The object A is compact in $\mathbf{D}(\mathcal{A})$ if and only if A has finite projective dimension.

(6.5.2) Let $\mathcal{A}$ be the module category of a commutative noetherian ring Λ. Show that a complex X in $\mathbf{D}(\mathcal{A})$ has finite cohomology if and only if $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(\Sigma^n C, X)$ is finitely generated over Λ for every compact object C and all $n \in \mathbb{Z}$, and if it vanishes for almost all $n \in \mathbb{Z}$.

(7.4.1) Let $\mathcal{A}$ be an additive category. Show that the two triangulated structures on $\mathbf{K}(\mathcal{A})$ (defined via mapping cones sequences and via degree-wise split exact sequences) coincide.

(7.4.2) Let Λ be a ring such that projective and injective Λ-modules coincide. Then Λ is noetherian and the category $\mathcal{A}$ of finitely generated Λ-modules is an abelian Frobenius category. Denote by $\mathbf{D}^b(\mathrm{Proj}\,\mathcal{A})$ the thick subcategory of $\mathbf{D}^b(\mathcal{A})$ which is generated by all projective modules. Show that the composition

$$\mathcal{A} \longrightarrow \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})/\mathbf{D}^b(\mathrm{Proj}\,\mathcal{A})$$

of canonical functors induces an equivalence $\mathbf{S}(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})/\mathbf{D}^b(\mathrm{Proj}\,\mathcal{A})$ of triangulated categories.

(7.5.1) Let $\mathcal{A}$ be a Frobenius category and $\tilde{\mathcal{A}}$ the full subcategory of acyclic complexes with injective components in $\mathbf{C}(\mathcal{A})$. Show that $\tilde{\mathcal{A}}$ is a Frobenius category (with respect to the degree-wise split exact sequences) and that the functor $\mathbf{S}(\tilde{\mathcal{A}}) \to \mathbf{S}(\mathcal{A})$ sending X to $Z^0 X$ is an equivalence.