K-theory and derived equivalences

(Joint work with D. Dugger)

Main Theorem:
If R and S are two derived equivalent rings, then $K_*(R) \cong K_*(S)$.

Neeman proved the above result for *regular rings*.

Main Example of a Quillen model category:

Ch_R : $\mathbb{Z}$-graded complexes of R-modules

weak equivalences: quasi-isomorphisms

fibrations: epimorphisms

cofibrations: not enough to be a monomorphism with projective cokernels. Any bounded complex of projectives is cofibrant.

$$\mathcal{H}o(\mathrm{Ch}_R) \;=\; \mathrm{Ch}_R[\text{q-iso}^{-1}] \;\cong_\Delta\; \mathcal{D}_R$$

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Definition: R and S are *derived equivalent* if $\mathcal{D}_R$ and $\mathcal{D}_S$ are equivalent as triangulated categories.

$$\mathcal{D}_R \cong_\Delta \mathcal{D}_S$$

Examples:

• Morita equivalences such as R and $M_n(R)$.

• Broué's Conjecture from the representation theory of finite groups provides other examples. For example the principal blocks of $k[A_4]$ and $k[A_5]$ (char $k = 2$).

A *stronger* notion of equivalence:

Definition: A *Quillen equivalence* between $\mathcal{M}$ and $\mathcal{N}$ is an adjoint pair of functors $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ such that

(1) L preserves cofibrations and R preserves fibrations.

(2) The derived functors $\overline{L} : \mathcal{H}o(\mathcal{M}) \rightleftarrows \mathcal{H}o(\mathcal{N}) : \overline{R}$ induce an equivalence of categories.

$$\mathcal{M} \simeq_Q \mathcal{N}$$

Examples:

- There exist two DGAs A and B which are derived equivalent, but there is no underlying Quillen equivalence between the model categories of differential graded modules over A and over B. In fact, $K_*(A) \not\cong K_*(B)$.

$$\mathcal{D}_A \cong_\Delta \mathcal{D}_B, \text{ but}$$
$$\text{d.g.}A\text{-mod} \not\simeq_Q \text{d.g.}B\text{-mod}$$

- $\mathcal{H}o(K(n)\text{-mod}) \cong_\Delta \mathcal{H}o(\text{d.g.}\mathbb{F}_p[v_n, v_n^{-1}]\text{-mod})$ but there is no underlying Quillen equivalence.

The *KEY* to our main result is the following:

Proposition 1: Ch_R and Ch_S are Quillen equivalent if and only if R and S are derived equivalent.

The main result then follows from a more rigorous version of the following loose statement.

"Proposition 2:" "A Quillen equivalence $\mathcal{M} \simeq_Q \mathcal{N}$ between stable model categories induces a K-theory equivalence, $K(\mathcal{M}) \simeq K(\mathcal{N})$."

Classical Morita Theory

Given an equivalence $F : \text{Mod-}S \rightleftarrows \text{Mod-}R : G$, consider $F(S) = P$.

It follows that

(i) $\text{Hom}_R(P, P) \cong \text{Hom}_S(S, S) \cong S$

(ii) If $\text{Hom}_R(P, X) \cong 0$ then $X \cong 0$.
(Since $\text{Hom}_R(P, X) \cong \text{Hom}_S(S, G(X))$.)
In this case P is a *strong generator.*

Morita Theory: The following are equivalent:

(1) Mod- R and Mod- S are equivalent categories.

(2) There exists a f. g. projective R-module P satisfying (i) and (ii).

(3) There are bimodules M and N such that ...

Proof: (1) implies (2) follows from the above.

(2) implies (1):
$\text{Hom}_R(P, -) : \text{Mod-}R \rightleftarrows \text{Mod-}S : - \otimes_S P$ is an equivalence.

Homotopical Morita Theory

Given a Quillen equivalence $L : \mathrm{Ch}_S \rightleftarrows \mathrm{Ch}_R : R$, consider $L(S) = P$.

It follows that

(i) $\mathcal{D}_R(P, P)_* \cong S$, concentrated in degree 0.

(ii) If $\mathcal{D}_R(P, X) \cong 0$, then $X \cong 0$.
In this case P is a *weak generator*.

(iii) $\oplus_i \mathcal{D}_R(P, X_i) \cong \mathcal{D}_R(P, \oplus_i X_i)$.
(Since $\mathcal{D}_S(S, -)$ also has this property.)
In this case P is *compact*.

Lemma: (Bökstedt-Neeman) In Ch_R, P is compact if and only if P is quasi-isomorphic to a bounded complex of finitely generated projectives. Thus, here we can assume P is such a bounded complex.

Proposition 1: The following are equivalent:

(1) $\mathrm{Ch}_R \simeq_Q \mathrm{Ch}_S$

(2) $\mathcal{D}_R \cong_\Delta \mathcal{D}_S$ (R and S are derived equivalent.)

(3) $(\mathcal{D}_R)_{\text{compact}} \cong_\Delta (\mathcal{D}_S)_{\text{compact}}$

(4) $K_b(\text{proj-}R) \cong_\Delta K_b(\text{proj-}S)$

(5) There is a bounded complex of finitely generated projectives P in Ch_R such that (i) and (ii) hold.
In this case P is a *tilting complex*.

Proposition 1: The following are equivalent:
(1) $\mathrm{Ch}_R \simeq_Q \mathrm{Ch}_S$
(2) $\mathcal{D}_R \cong_\Delta \mathcal{D}_S$ (R and S are derived equivalent.)
(3) $(\mathcal{D}_R)_{\text{compact}} \cong_\Delta (\mathcal{D}_S)_{\text{compact}}$
(4) $K_b(\text{proj-}R) \cong_\Delta K_b(\text{proj-}S)$
(5) There is a tilting complex P in Ch_R.

Rickard showed (4) and (5) are equivalent and also established several other statements equivalent to (4). But, even the equivalence of (2) and (3) is new here.

This is implicitly in Schwede-S.; that proof involves tilting spectra.

Proof: (1) $\to$ (2) $\to$ (3) $=$ (4) $\to$ (5) as above.

(5) $\to$ (1): Let $\mathcal{E}(P) = \mathrm{Hom}_R(P, P)$. Then

$$\mathrm{Hom}_R(P, -) : \mathrm{Ch}_R \rightleftarrows \text{Mod-}\,\mathcal{E}(P) : - \otimes_{\mathcal{E}(P)} P$$

is a Quillen equivalence.

$\mathcal{E}(P)$ is quasi-iso to S (since $H_*\mathcal{E}(P) = S$). Hence,

$$\mathrm{Ch}_R \ \simeq_Q \ \text{d.g. Mod-}\,\mathcal{E}(P) \ \simeq_Q \ \text{d.g. Mod-}\,S \ = \ \mathrm{Ch}_S$$

K-theory and Proposition 2

Definition:

(1) A *Waldhausen subcategory* $\mathcal{U}$ of a pointed model category $\mathcal{M}$ is a full subcategory of cofibrant objects containing $*$ such that if $A \rightarrowtail B$ and $A \to X$ are in $\mathcal{U}$ then $(A \coprod_B X)_{\text{in } \mathcal{M}}$ is also in $\mathcal{U}$.

(2) $\overline{\mathcal{U}}$ is the full subcategory of cofibrant objects in $\mathcal{M}$ weakly equivalent to an object in $\mathcal{U}$.

(3) If $\mathcal{U} \cong \overline{\mathcal{U}}$ then we say $\mathcal{U}$ is *complete*.

Main Example: $\mathcal{M} = \mathrm{Ch}_R$ and $\mathcal{U}_R$ contains the bounded complexes of f. g. projectives.
Thus, $\overline{\mathcal{U}} = \mathcal{M}_{\text{compact, cofibrant}}$.

Lemma: Any Waldhausen subcategory is a "category of cofibrations and weak equivalences" in the sense of Waldhausen.

Definition: $K(\mathcal{U})$ is the Waldhausen K-theory of $\mathcal{U}$. (Using the S-dot construction.)
Note, $K(\mathcal{U}_R) \cong K(R)$.

Proposition 2: Assume $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is a Quillen equivalence and $\mathcal{U}$ is a complete Waldhausen subcategory of $\mathcal{M}$. Then

- $\mathcal{V} = \overline{L\mathcal{U}}$ is a complete Waldhausen subcategory of $\mathcal{N}$ and
- $L : K(\mathcal{U}) \xrightarrow{\simeq} K(\mathcal{V})$.

Corollary: If $L : \mathrm{Ch}_R \rightleftarrows \mathrm{Ch}_S : R$ is a Quillen equivalence, then $K(R) \xrightarrow{\simeq} K(S)$.

Proof: One can show that: $\overline{L(\overline{\mathcal{U}_R})} = \overline{\mathcal{U}_S}$. The forward inclusion follows since L preserves compact and cofibrant objects. For the other direction, if $X \in \overline{\mathcal{U}_S}$, then $\overline{R}(X) \in \overline{\mathcal{U}_R}$. Since $L(\overline{R}(X)) \simeq X$ it follows that $X \in \overline{L(\overline{\mathcal{U}_R})}$.

Proposition 2 is similar to one of Thomason's results, but for categories of chain complexes it requires functors to be prolongations from modules. So his result doesn't apply to tilting complexes in general.

Since R and S are derived equivalent if and only if they are Quillen equivalent, the main result that derived equivalence implies K-theory equivalence follows from the corollary.

Lemma: Let w$\mathcal{U}$ be the subcategory of weak equivalences in $\mathcal{U}$. Then $N(w\mathcal{U}) \xrightarrow{NL} N(w\mathcal{V})$ is a homotopy equivalence with inverse $N(\overline{R})$.

Proof of Proposition 2: Each level of the S-dot construction is a weak equivalence. This follows from the lemma. At level n, considering $w\mathcal{U}_n$ with $\mathcal{U}_n$ the cofibrant diagrams $A_0 \to A_1 \to \cdots A_n$ such that each A_i is in $\mathcal{U}$.

General result: Assume $\mathcal{A}$ and $\mathcal{B}$ are cocomplete abelian categories with sets of small projective *strong* generators.

(1) $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only if $\mathrm{Ch}_{\mathcal{A}}$ and $\mathrm{Ch}_{\mathcal{B}}$ are Quillen equivalent.
(2) If $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent, then $K_c(\mathcal{A}) \simeq K_c(\mathcal{B})$ (where K_c is the K-theory of the compact objects.)

Neeman considered instead small abelian categories and the associated Quillen K-theory of exact categories.

Example: There exist two DGAs A and B such that

$$\mathcal{D}_A \cong_\Delta \mathcal{D}_B, \text{ but}$$
$$\text{d.g.}A\text{-mod} \not\simeq_Q \text{d.g.}B\text{-mod and}$$
$$K_*(A) \ncong K_*(B)$$

This example is based on Marco Schlichting's example. He showed that the stable module categories of $\mathbb{Z}/p^2$ and $\mathbb{Z}/p[\epsilon]/\epsilon^2$ have equivalent triangulated homotopy categories but different K-theories for $p > 3$.

One can find DGAs A and B such that:

- Stmod($\mathbb{Z}/p[\epsilon]/\epsilon^2$) $\simeq_Q$ d.g.A-mod
- Stmod($\mathbb{Z}/p^2$) $\simeq_Q$ d.g.B-mod

Here A and B are the endomorphism DGAs of the Tate resolution of a generator ($\mathbb{Z}/p$ in both cases):

- $A = \mathbb{Z}/p[x_1, x_1^{-1}]$ with $d = 0$.
- $B = \mathbb{Z}[x_1, x_1^{-1}]\langle e_1\rangle/e^2 = 0, ex + xe = x^2$ with $de = p$ and $dx = 0$.