An introduction to stable homotopy theory

"Abelian groups up to homotopy"
spectra $\Longleftrightarrow$ generalized cohomology theories

Examples:

1. **Ordinary cohomology:**

For A any abelian group, $H^n(X;A) = [X_+, K(A,n)]$.

Eilenberg-Mac Lane spectrum, denoted HA.
$HA_n = K(A,n)$ for $n \geq 0$.

The coefficients of the theory are given by

$$HA^*(\mathrm{pt}) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

2. **Hypercohomology:**
For $C.$ any chain complex of abelian groups,
$\mathbb{H}^s(X; C.) \cong \oplus_{q-p=s} H^p(X; H_q(C.))$.
Just a direct sum of shifted ordinary cohomologies.

$HC.^*(\text{pt}) = H_*(C.)$.

3. **Complex K-theory:**
$K^*(X)$; associated spectrum denoted K.

$$K_n = \begin{cases} U & n = \text{odd} \\ BU \times \mathbb{Z} & n = \text{even} \end{cases}$$

$$K^*(\text{pt}) = \begin{cases} 0 & * = \text{odd} \\ \mathbb{Z} & * = \text{even} \end{cases}$$

4. **Stable cohomotopy:**
$\pi_S^*(X)$; associated spectrum denoted $\mathbb{S}$.

$\mathbb{S}_n = S^n$, $\mathbb{S}$ is the *sphere spectrum.*

$\pi_S^*(\text{pt}) = \pi_{-*}^S(\text{pt})$ = stable homotopy groups of spheres. These are only known in a range.

"Rings up to homotopy"
ring spectra $\Longleftrightarrow$gen. coh. theories with a product

1. For R a ring, HR **is a ring spectrum.**
The cup product gives a graded product:
$HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$

Induced by $K(R,p) \wedge K(R,q) \to K(R,p+q)$.

2. For $A.$ a differential graded algebra (DGA),
$HA.$ **is a ring spectrum.** Product induced by
$\mu : A. \otimes A. \to A.$, or $A_p \otimes A_q \to A_{p+q}$.

The groups $\mathbb{H}(X; A.)$ are still determined by $H_*(A)$,
but the product structure is *not* determined $H_*(A)$.

3. K **is a ring spectrum;**
Product induced by tensor product of vector bundles.

4. $\mathbb{S}$ **is a commutative ring spectrum.**

Definition. A "*ring spectrum*" is a sequence of pointed spaces $R = (R_0, R_1, \cdots, R_n, \cdots)$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

Definition. A "*spectrum*" F is a sequence of pointed spaces $(F_0, F_1, \cdots, F_n, \cdots)$ with structure maps $\Sigma F_n \to F_{n+1}$. Equivalently, adjoint maps $F_n \to \Omega F_{n+1}$.

Example: $\mathbb{S}$ a commutative ring spectrum

Structure maps: $\Sigma S^n = S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$.

Product maps: $S^p \wedge S^q \xrightarrow{\cong} S^{p+q}$.

Actually, must be more careful here. For example: $S^1 \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge S^1$ is a degree -1 map.

History of spectra and $\wedge$

Boardman in 1965 defined spectra and $\wedge$. $\wedge$ is only commutative and associative up to homotopy.

A_∞ *ring spectrum* = best approximation to associative ring spectrum.

E_∞ *ring spectrum* = best approximation to commutative ring spectrum.

Lewis in 1991: No good $\wedge$ exists.
Five reasonable axioms $\Longrightarrow$ no such $\wedge$.

Since 1997, lots of monoidal categories of spectra exist! (with $\wedge$ that is commutative and associative.)
1. 1997: Elmendorf, Kriz, Mandell, May
2. 2000: Hovey, S., Smith
3, 4 and 5 ... Lydakis, Schwede, ...

Theorem.(Mandell, May, Schwede, S. '01; Schwede '01)
All above models define the same homotopy theory.

Spectral Algebra

Given the good categories of spectra with $\wedge$, one can easily do algebra with spectra.

Definitions:

A *ring spectrum* is a spectrum R with an associative and unital multiplication $\mu : R \wedge R \to R$ (with unit $\mathbb{S} \to R$).

An *R-module spectrum* is a spectrum M with an associative and unital action $\alpha : R \wedge M \to M$.

$\mathbb{S}$-*modules* are spectra.
$S^1 \wedge F_n \to F_{n+1}$ iterated gives $S^p \wedge F_q \to F_{p+q}$.
Fits together to give $\mathbb{S} \wedge F \to F$.

$\mathbb{S}$-*algebras* are ring spectra.

Homological Algebra vs. Spectral Algebra

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	$\mathbb{S}$
$\mathbb{Z}$-Mod $= \mathcal{A}b$	d.g.-Mod $= \mathcal{C}h$	$\mathbb{S}$-Mod $= \mathcal{S}pectra$
$\mathbb{Z}$-Alg $=$ $\mathcal{R}ings$	d.g.-Alg $=$ $\mathcal{D}GAs$	$\mathbb{S}$-Alg $=$ $\mathcal{R}ing\ spectra$

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	H$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-Mod	d.g.-Mod	H$\mathbb{Z}$-Mod	$\mathbb{S}$-Mod
$\mathbb{Z}$-Alg	d.g.-Alg	H $\mathbb{Z}$-Alg	$\mathbb{S}$-Alg
$\cong$	quasi-iso	weak equiv.	weak equiv.

Quasi-isomorphisms are maps which induce isomorphisms in homology.

Weak equivalences are maps which induce isomorphisms on the coefficients.

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	H$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-Mod	d.g.-Mod	H$\mathbb{Z}$-Mod	$\mathbb{S}$-Mod
$\mathbb{Z}$-Alg	d.g.-Alg	H $\mathbb{Z}$-Alg	$\mathbb{S}$-Alg
$\cong$	quasi-iso	weak equiv.	weak equiv.
	$\mathcal{D}(\mathbb{Z}) =$ $\mathcal{C}h[\text{q-iso}]^{-1}$	$\mathcal{H}o(H\mathbb{Z}\text{-Mod})$	$\mathcal{H}o(\mathcal{S}) =$ $\mathcal{S}pectra[\text{wk.eq.}]^{-1}$

Theorem. (Robinson '87; Schwede-S. '03; S. '07) Columns two and three are equivalent up to homotopy.

(1) $\mathcal{D}(\mathbb{Z}) \simeq_{\Delta} \mathcal{H}o(H\mathbb{Z}\text{-Mod})$.

(2) $\mathcal{C}h \simeq_{\text{Quillen}} H\mathbb{Z}\text{-Mod}$.

(3) Associative $\mathcal{D}GA \simeq_{\text{Quillen}}$ Assoc. $H\mathbb{Z}$-Alg.

(4) For $A.$ a DGA,
d.g. $A.$-Mod $\simeq_{\text{Quillen}} HA.$-Mod
and $\mathcal{D}(A.) \simeq_{\Delta} \mathcal{H}o(HA.\text{-Mod})$.

Algebraic Models

Thm.(Gabriel)
Let $\mathcal{C}$ be a cocomplete, abelian category with a small projective generator G. Let $\mathcal{E}(G) = \mathcal{C}(G, G)$ be the endomorphism ring of G. Then

$$\mathcal{C} \cong \text{Mod-}\mathcal{E}(G)$$

Consider $\mathcal{C}(G, -): \; X \to \mathcal{C}(G, X)$.

Spectral model categories

Defn: Let Sp denote a monoidal model category of spectra. $\mathcal{C}$ is a Sp-*model category* if it is compatibly enriched and tensored over Sp. $\mathcal{E}(X) = F_{\mathcal{C}}(X, X)$ is a ring spectrum.

Thm: (Schwede-S.) If $\mathcal{C}$ is a Sp-model category with a (cofibrant and fibrant) small generator G then $\mathcal{C}$ is Quillen equivalent to (right) module spectra over $\mathcal{E}(G) = F_{\mathcal{C}}(G, G)$.

$$\mathcal{C} \simeq_Q \text{Mod-}\,\mathcal{E}(G)$$

$$- \otimes_{\mathcal{E}(G)} G : \text{Mod-}\,\mathcal{E}(G) \rightleftarrows \mathcal{C} : F_{\mathcal{C}}(G, -)$$

Rational stable model categories

Defn: A Sp-model category is rational if $[X, Y]_{\mathcal{C}}$ is a rational vector space for all X, Y in $\mathcal{C}$. In this case $\mathcal{E}(X) = F_{\mathcal{C}}(X, X) \simeq H\mathbb{Q} \wedge cF_{\mathcal{C}}(X, X)$.

Rational spectral algebra $\simeq$ d.g. algebra:

- There are composite Quillen equivalences
$$\Theta : H\mathbb{Q}\text{-Alg} \rightleftarrows \mathrm{DGA}_{\mathbb{Q}} : \mathbb{H}.$$
- For any $H\mathbb{Q}$-algebra spectrum B,
$$\text{Mod-}B \rightleftarrows \text{Mod-}\Theta B.$$

Thm: If $\mathcal{C}$ is a rational Sp-model category with a (cofibrant and fibrant) small generator G then there are Quillen equivalences:

$$\mathcal{C} \simeq_Q \text{Mod-}\mathcal{E}(G)$$

$$\simeq_Q \text{Mod-}(H\mathbb{Q} \wedge c\mathcal{E}(G))$$

$$\simeq_Q d.g.\,\text{Mod-}\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G)).$$

$\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G))$ is a rational dga with
$$H_*\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G)) \cong \pi_* c\mathcal{E}(G).$$