Model Category Theory

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This material comes from many sources (in particular: Quillen, Dwyer-Spalinski, Hovey, Goerss).

These slides are full of small lies, some of which are intentional.

Topic I: Definitions and Examples

First Example: Homological Algebra

An R-module P is **projective** if and only if given $f\colon A \to B$ surjective and $g\colon P \to B$ there is a lift $l\colon P \to A$ such that $fl = g$:

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \overset{l}{\nearrow} & \downarrow f \\ P & \underset{g}{\longrightarrow} & B \end{array}$$

We say $0 \to P$ has the **left lifting property** with respect to all surjections f.

Similarly, f is surjective if and only if for every projective P (and $g\colon P \to B$) a lift l exists such that $fl = g$.

We say f has the **right lifting property** (RLP) with respect to any map $0 \to P$ with P projective.

Chain complexes: $\mathcal{C}h_R$, non-negatively graded chain complexes of R-modules.

Lifting Property: Assume $i\colon A. \to B.$ is a monomorphism such that each B_k/A_k is projective and p is surjective in each degree $k > 0$:

$$\begin{array}{ccc} A. & \xrightarrow{f} & X. \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B. & \xrightarrow[g]{} & Y. \end{array}$$

then a lift l exists (with $li = f$ and $pl = g$) if either i or p is a **quasi-isomorphism** (i.e., induces an isomorphism in homology).

Factorization: Any map in $\mathcal{C}h_R$, $f\colon X. \to Y.$ factors in two ways:

(1) $X. \overset{i}{\rightarrowtail} Y.' \overset{p}{\twoheadrightarrow} Y.$ where i is a monomorphism with projective cokernels and p is a quasi-isomorphism and surjective.

(2) $X. \overset{j}{\rightarrowtail} X.' \overset{q}{\twoheadrightarrow} Y.$ where j is a monomorphism with projective cokernels and a quasi-isomorphism and q is surjective.

Second Example: Topological Spaces, $\mathcal{T}op$

The analogues of projective objects and surjective maps are CW complexes and fibrations.

Definition: A map of spaces $p\colon\ X \to Y$ is a **Serre fibration** if and only if p has the right lifting property with respect to inclusions $i\colon\ A \times 0 \to A \times [0,1]$ for each CW complex A.

$$\begin{array}{ccc} A \times 0 & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{l}{\nearrow} & \big\downarrow{\scriptstyle p} \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

Definition: Suppose given a direct system of inclusions of spaces $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ such that each pair (X_{n+1}, X_n) is a relative CW pair. Then we say the map $X_0 \to \operatorname{colim}_n X_n$ is a **generalized relative CW inclusion**.

Lifting Property: Assume $i\colon A \to B$ is a retract of a generalized relative CW inclusion and $p\colon X \to Y$ is a Serre fibration.

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{l}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

then a lift l exists if either i or p is a **weak equivalence** (induces an isomorphism on homotopy).

Factorization: Any map in $\mathcal{T}op$, $f\colon X \to Y$ factors in two ways:

(1) $X \overset{i}{\rightarrowtail} Y' \overset{p}{\twoheadrightarrow} Y$ where i is a generalized relative CW inclusion and p is a weak equivalence and a Serre fibration.

(2) $X. \overset{j}{\rightarrowtail} X' \overset{q}{\twoheadrightarrow} Y.$ where j is a generalized relative CW inclusion and a weak equivalence and q is a Serre fibration.

Definition: A **model category** is a category $\mathcal{C}$ with three distinguished classes of maps:

(1) weak equivalences ($\xrightarrow{\sim}$)
(2) cofibrations ($\rightarrowtail$)
(3) fibrations ($\twoheadrightarrow$)

each closed under composition and containing the identity maps and subject to the following axioms.

An **acyclic cofibration** ($\overset{\sim}{\rightarrowtail}$) is a cofibration which is a weak equivalence.

An **acyclic fibration** ($\overset{\sim}{\twoheadrightarrow}$) is a fibration which is a weak equivalence.

Axioms:

M1. $\mathcal{C}$ is closed under finite limits and colimits;

M2. (2 out of 3) If f and g are composable maps such that any two of the three maps f, g, gf are weak equivalences, then so is the third;

M3. (Retracts) The three distinguished classes of maps are closed under retracts;

M4. (Lifting) A lift l exists in every diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration, p is a fibration and i or p is a weak equivalence.

M5. (Factorization) Any map f can be factored in two ways:
(1) f = pi, where i is a cofibration and p is an acyclic fibration, and
(2) f = qj where j is an acyclic cofibration and q is a fibration.

Remarks:

M4 and M5 both have two parts.

M1 and M5 have variations.

$\mathcal{C}^{op}$ is also a model category.

By MC1, any model category has an initial object $\emptyset$ and a terminal object $*$.

Definitions:

An object X is **cofibrant** if $\emptyset \to X$ is a cofibration; Y is **fibrant** if $Y \to *$ is a fibration.

A **cofibrant replacement** cX exists for any X by the factorization axiom: $\emptyset \to X$ factors as $\emptyset \rightarrowtail cX \overset{\sim}{\twoheadrightarrow} X$.

Similarly a **fibrant replacement** exists for any Y: due to the factorization $Y \overset{\sim}{\rightarrowtail} fY \twoheadrightarrow *$.

Lemma: The three classes of maps are not independent:

A map is a cofibration if and only if it has the LLP (left lifting property) with respect to any acyclic fibration.

A map is an acyclic cofibration if and only if it has the LLP with respect to any fibration.

Similarly, the fibrations and acyclic fibrations can be defined using RLPs (right lifting properties).

Examples:

Projective model category: $\mathcal{C}h_R^{proj}$
(1) The *weak equivalences* are the quasi-isomorphisms.
(2) The *fibrations* are the maps which are surjective in positive degrees.
(3) The *cofibrations* are the monomorphisms with levelwise projective cokernels.

All complexes are fibrant here. The cofibrant replacement of a module is a projective resolution.

Injective model category: $\mathcal{C}h_R^{inj}$
(1) The *weak equivalences* are the quasi-isomorphisms.
(2) The *fibrations* are the maps which are surjective with levelwise injective kernels.
(3) The *cofibrations* are the monomorphisms.

All complexes are cofibrant here. The fibrant replacement of a module is an injective resolution.

Two different model structures on the same underlying category with the same weak equivalences.

Next, two different model structures on the same category with different weak equivalences.

Weak equiv. model structure: $\mathcal{T}op^{w.e.}$
(1) *Weak equivalences* the maps inducing isomorphisms in homotopy.
(2) *Fibrations* the Serre fibrations.
(3) *Cofibrations* the retracts of generalized relative CW inclusions.

Homotopy equiv. model structure: $\mathcal{T}op^{h.e.}$
(1) *Weak equivalences* the homotopy equivalences.
(2) *Fibrations* the Hurewicz fibrations.
(3) *Cofibrations* the closed Hurewicz cofibrations.

A map $p\colon X \to Y$ is a **Hurewicz fibration** if p has the LLP with respect to $A \times 0 \to A \times [0,1]$ for *every* space A.

An inclusion of a closed subspace $i\colon A \to B$ is a **closed Hurewicz cofibration** if a lift exists in every diagram below for every space Y:

$$\begin{array}{ccc} B \times 0 \cup A \times [0,1] & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B \times [0,1] & \longrightarrow & * \end{array}$$

Topic II: Brief introduction to $s\mathcal{S}et$

A simplicial set $X.$ is a sequence of sets X_n with face maps $d_i\colon X_n \to X_{n-1}$ and degeneracy maps $s_j\colon X_n \to X_{n+1}$ for $0 \leq i, j \leq n$ such that certain *simplicial identities* hold among composites of these maps.

Example: The **standard n-simplex** $\Delta[n]$ has $(\Delta[n])_q = \{(a_0, \cdots, a_q) | 0 \leq a_0 \leq \cdots \leq a_q \leq n\}$ with $d_i(a_0, \cdots, a_q) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots a_q)$ and $s_j(a_0, \cdots a_q) = (a_0, \cdots, a_j, a_j, \cdots a_q)$.

$\Delta[n]$ has exactly one non-degenerate n-simplex: $(0, 1, \cdots, n) = \iota_n$. Every other simplex is the image of ι_n under some composite of d_i and s_j maps.

Each n-simplex of a simplicial set $X.$ corresponds to a map from $\Delta[n]$: $X_n = s\mathcal{S}et(\Delta[n], X.)$.

Formally, the category of simplicial sets is the category of contravariant functors from a category $\mathbf{\Delta}$ to sets.

$$s\mathcal{S}et = (\mathcal{S}et)^{\mathbf{\Delta}^{op}}$$

$\mathbf{\Delta}$ has objects $[n] = \{0, 1, \cdots, n\}$ and morphisms the (weakly) order preserving maps (which are all composites of maps d^i, which skips i, and s^j, which repeats j).

Examples: We see that $\Delta[n] = \mathbf{\Delta}(-, [n])$ and d^i in $\mathbf{\Delta}$ induces a map $d^i \colon \Delta[n-1] \to \Delta[n]$.

Define the *boundary* $\partial\Delta[n] \subseteq \Delta[n]$:

$$\partial\Delta[n] = \cup_{0 \le i \le n} d^i \Delta[n-1]$$

and the *horn* $\Delta^k[n] \subseteq \Delta[n]$:

$$\Delta^k[n] = \cup_{i \neq k} d^i \Delta[n-1]$$

Adjoint functors: $|-|\colon s\mathcal{S}et \rightleftarrows \mathcal{T}op\colon \operatorname{Sing}$

There are *topological standard n-simplices* σ_n with maps $d^i\colon \sigma_{n-1} \to \sigma_n$ and $s^j\colon \sigma_{n+1} \to \sigma_n$ which satisfy the dual of the simplicial identities.

Define the **geometric realization** of a simplicial set $X.$ by $|X.| = (\cup_n X_n \times \sigma_n)/(d_i x, u) \sim (x, d^i u)$.

$|X.|$ is a CW-complex with one n-cell for each non-degenerate n-simplex of X. (E.g. $|\Delta[n]| = \sigma_n$.)

The right adjoint of $|-|\colon s\mathcal{S}et \to \mathcal{T}op$ is the **singular set** functor $\operatorname{Sing}\colon \mathcal{T}op \to s\mathcal{S}et$.

$$(\operatorname{Sing}(X))_n = \mathcal{T}op(\sigma_n, X)$$

Model category for $s\mathcal{S}et$:
(1) A map f is a *weak equivalence* if $|f|$ is a weak equivalence of spaces.
(2) The *cofibrations* are the monomorphisms.
(3) The *fibrations* are the maps with the RLP with respect to $\Delta^k[n] \to \Delta[n]$, for all k, n.

In fact, i is a cofibration in $s\mathcal{S}et$ if and only if $|i|$ is a cofibration in $\mathcal{T}op$.

Dually, $\operatorname{Sing} p$ is a fibration in $s\mathcal{S}et$ if and only if p is a Serre fibration in $\mathcal{T}op$.

The acyclic fibrations are the maps with the RLP with respect to $\partial\Delta[n] \to \Delta[n]$ for all n.

Topic III: The Homotopy Category

Definition: For the homotopy category $\mathcal{H}o(\mathcal{C})$

$$\text{objects}(\mathcal{H}o(\mathcal{C})) = \text{objects}(\mathcal{C})$$

$$\mathcal{H}o(\mathcal{C})(X, Y) = \{X \to X_1 \overset{\sim}{\leftarrow} X_2 \to \cdots \overset{\sim}{\leftarrow} Y\}/\sim$$

Example:

$$\mathcal{H}o(\mathcal{C}h_R^{proj})(M, N) \cong \mathcal{C}h_R(P.(M), N)/\sim \text{(chain homotopy)}$$

where $P.(M)$ is a projective resolution of M (or a cofibrant replacement).

Chain homotopy: Let $I = R[1] \oplus R[0] \oplus R[0]$ with boundary $\partial(x, a, b) = (0, x, -x)$.
$(P. \otimes I) = \Sigma P. \oplus P. \oplus P.$ with a natural inclusion $i\colon P. \oplus P. \to P. \otimes I$.

A **chain homotopy** between $f, g\colon P. \to Y.$ is a map H which completes the following diagram:

$$\begin{array}{ccc} P. \oplus P. & \longrightarrow & P. \otimes I \\ & \searrow^{f \oplus g} & \vdots\, H \\ & & Y. \end{array}$$

Definition: Let A be an object in a model category $\mathcal{C}$. A **cylinder object** for A is a factorization of the fold map $\nabla = qi$

$$\begin{array}{ccc} A \oplus A & \overset{i}{\rightarrowtail} & C(A) \\ & \underset{\nabla}{\searrow} & \downarrow{\scriptstyle \sim}\, q \\ & & A \end{array}$$

such that i is a cofibration and q is a weak equivalence.

Example: In $\mathcal{C}h_R$, for $P.$ cofibrant $C(P.) = P. \otimes I$. In $\mathcal{T}op$, for A a CW complex, $C(A) = A \times [0, 1]$.

Definition: Assume A is cofibrant. A **left homotopy** between $f, g\colon A \to Y$ is a diagram:

$$\begin{array}{ccc} A \coprod A & \longrightarrow & C(A) \\ & \underset{f \coprod g}{\searrow} & \vdots\, H \\ & & Y. \end{array}$$

where $C(A)$ is a cylinder object for A.

Definition: Let X be an object in a model category $\mathcal{C}$. A **path object** for X is a factorization of the diagonal $\Delta = pj$

$$\begin{array}{ccc} X & \xrightarrow[j]{\sim} & X^I \\ & \searrow^{\Delta} & \downarrow p \\ & & X \times X \end{array}$$

such that j is a weak equivalence and p is a fibration.

Examples: In $\mathcal{C}h_R$, $(X.)^I = \mathrm{Hom}_{\mathcal{C}h_R}(I, X.)$.
In $\mathcal{T}op$, $X^I = X^{[0,1]}$ is a path object.

Definition: Assume X is fibrant. A **right homotopy** between $f, g\colon\ B \to X$ is a diagram:

$$\begin{array}{ccc} & & X^I \\ & \overset{H}{\nearrow} & \downarrow p \\ B & \xrightarrow[f \times g]{} & X \times X \end{array}$$

where X^I is a path object for X.

Lemma: If A is cofibrant and X is fibrant, then f, g are left homotopic if and only if they are right homotopic. We write $f \sim g$.

For each object X in a model category $\mathcal{C}$ fix a cofibrant and a fibrant replacement: cX and fX.

Theorem: The homotopy category of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and with

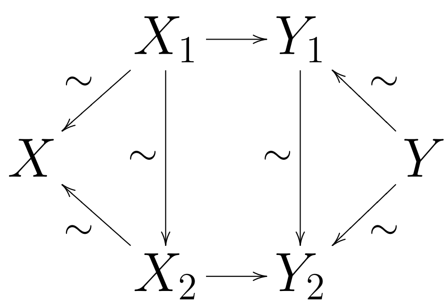
$$\mathcal{H}o(\mathcal{C})(X,Y) \cong [fcX, fcY]_{\mathcal{C}} \cong [cX, fY]_{\mathcal{C}}$$

where $[X,Y]_{\mathcal{C}} = \mathcal{C}(X,Y)/\sim$.

Lemma: (Dwyer-Kan)

$$\mathcal{H}o(\mathcal{C})(X,Y) \cong \{X \xleftarrow{\sim} X' \to Y' \xleftarrow{\sim} Y\}/\sim$$

where the equivalence relation is generated by diagrams of the form



Topic IV: Derived Functors

A **Quillen functor** from $\mathcal{C}$ to $\mathcal{D}$ (two model categories) is an adjoint pair of functors:

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$$

such that the left adjoint F preserves cofibrations and acyclic cofibrations. (It follows that G preserves fibrations and acyclic fibrations.)

Ken Brown's Lemma: Any left Quillen functor F takes weak equivalences between cofibrant objects to weak equivalences. Dually, G takes weak equivalences between fibrant objects to weak equivalences.

Any Quillen functor $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ induces adjoint **total derived functors**

$$LF\colon \mathcal{H}o(\mathcal{C}) \rightleftarrows \mathcal{H}o(\mathcal{D})\colon RG$$

defined by $LF(X) = F(cX)$ and $RG(Y) = G(fY)$.

Example: $- \otimes N$ is a left Quillen functor. $\mathrm{Tor}_i^R(M,N) = H_i(L(-\otimes_R N)) = H_i(P.(M) \otimes_R N)$

Example: Homotopy colimits, homotopy limits.

Definition: A Quillen functor F is a **Quillen equivalence** if and only if the induced adjoint pair $LF\colon \mathcal{H}o(\mathcal{C}) \rightleftarrows \mathcal{H}o(\mathcal{D})\colon RG$ is an equivalence of categories.

Equivalently, F is a *Quillen equivalence* if for each cofibrant A in $\mathcal{C}$ and each fibrant X in $\mathcal{D}$ a map $f\colon FA \xrightarrow{\sim} X$ is a weak equivalence in $\mathcal{D}$ if and only if its adjoint $f'\colon A \xrightarrow{\sim} GX$ is a weak equivalence in $\mathcal{C}$.

Example: There is a Quillen functor

$$id\colon \mathcal{C}h_R^{proj} \rightleftarrows \mathcal{C}h_R^{inj}\colon id$$

which is a Quillen equivalence.

$$\mathcal{H}o(\mathcal{C}h_R^{proj}) \cong \mathcal{H}o(\mathcal{C}h_R^{inj}) \cong \mathcal{D}(R)$$

Example: There is a Quillen functor

$$id\colon \mathcal{T}op^{w.e.} \rightleftarrows \mathcal{T}op^{h.e.}\colon id$$

which is not a Quillen equivalence.

$\mathcal{H}o(\mathcal{T}op^{w.e.})$ = homotopy category of CW-complexes.
$\mathcal{H}o(\mathcal{T}op^{h.e.})$ = homotopy category of all topological spaces.

Theorem: $|-|: \ s\mathcal{S}et \rightleftarrows \mathcal{T}op^{w.e.} :$ Sing is a Quillen equivalence.

Theorem: ([Quillen], [B-G])

$$\mathcal{H}o(s\mathcal{S}et_{1,\mathbb{Q},f})^{op} \cong \mathcal{H}o(CDGA_{1,\mathbb{Q},f})$$

$CDGA_{1,\mathbb{Q}}$ = simply connected, finite type, commutative dg $\mathbb{Q}$-algebras
$s\mathcal{S}et_{1,\mathbb{Q}}$ = simply connected, finite type, weak equivalences are $H_*(-,\mathbb{Q})$ isomorphisms.
Finite type = each $H^k(-,\mathbb{Q})$ is finitely generated.

Theorem: [Mandell '01]

$$\mathcal{H}o(\mathcal{T}op_{1,p-complete,f.p-type})^{op} \cong \text{full subcat.}\mathcal{H}o(E_\infty DGA_{\overline{\mathbb{F}_p}})$$

Theorem: [Mandell '03]
$X \xrightarrow{\sim} Y$ if and only if $C^*(X) \xrightarrow{\sim} C^*(Y)$
(for X, Y finite type, simply-connected.)

$$\mathcal{H}o(\mathcal{T}op_{1,f})^{op} \xrightarrow{\text{faithful}} \mathcal{H}o(E_\infty DGA_{\mathbb{Z}})$$

Topic V: Cofibrantly generated model categories

Example: $\mathcal{T}op^{w.e.}$ is cofibrantly generated.
(1) f is a *fibration* if and only if it has the RLP with respect to $J = \{D^n \times 0 \to D^n \times [0,1]\}$.
(2)f is an *acyclic fibration* if and only if it has the RLP with respect to $I = \{S^{n-1} \to D^n\}$.
(3) Any *(acyclic) cofibration* is in I-cof (J-cof).

I-cell denotes the maps built from I using pushouts and possibly infinite compositions (colimits).
I-cof denotes the retracts of maps in I-cell.

Example: In $\mathcal{C}h_R^{proj}$, let $S^n = R[n]$ and $D^n = R[n+1] \oplus R[n]$ with $d_{n+1} = id$. Then (1)-(3) above hold with $I = \{S^{n-1} \to D^n\}$ and $J = \{0 \to D^n\}$.

A is **sequentially small** if there is a bijection.

$$\operatorname*{colim}_n \mathcal{C}(A, B_n) \to \mathcal{C}(A, \operatorname*{colim}_n B_n)$$

In $\mathcal{C}h_R$, S^n and D^n are sequentially small. In $\mathcal{T}op$, S^n and D^n are sequentially small with respect to cofibrations (each $B_n \rightarrowtail B_{n+1}$ is a cofibration).

Small object argument: Suppose the domains of K are sequentially small with respect to maps in K-cell. Given any map $X \xrightarrow{f} Y$, there is a functorial factorization $X \xrightarrow{i_\infty(f)} Z \xrightarrow{p_\infty(f)} Y$ such that $i_\infty(f)$ is in K-cell and $p_\infty(f)$ has the RLP with respect to K.

Proof: We'll inductively construct a sequence

$$\begin{array}{ccccccccc} X = Z_0 & \xrightarrow{i_0} & Z_1 & \xrightarrow{i_1} & Z_2 & \longrightarrow \cdots \longrightarrow & Z = \operatorname{colim} Z_n \\ & {\scriptstyle f=p_0} \searrow & \downarrow {\scriptstyle p_1} & \swarrow {\scriptstyle p_2} & & \swarrow {\scriptstyle p_\infty} & \\ & & Y & & & & \end{array}$$

Assume Z_n, p_n have been constructed. Let S_n be the set of commutative squares with $k \in K$:

$$\begin{array}{ccc} A & \longrightarrow & Z_n \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle p_n} \\ B & \longrightarrow & Y \end{array}$$

Define Z_{n+1} to be the pushout in the digram below

$$\begin{array}{ccc} \coprod_{s \in S_n} A_s & \longrightarrow & Z_n \\ {\scriptstyle k_s}\downarrow & & \downarrow{\scriptstyle i_n} \\ \coprod_{s \in S_n} B_s & \longrightarrow & Z_{n+1} \end{array}$$

The map p_{n+1} is induced by the map p_n.

Define $i_\infty(f)\colon X \to Z$ to be the composition of the maps i_n and $p_\infty(f) = \operatorname{colim} p_n$. It follows that $i_\infty(f)$ is in K-cell.

Check that $p_\infty(f)$ has the RLP w.r.t. K: Given

$$\begin{array}{ccc} A & \xrightarrow{g} & Z \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle p_\infty(f)} \\ B & \xrightarrow[h]{} & Y \end{array}$$

Since A is sequentially small w.r.t. K-cell, g factors through some Z_j

$$\begin{array}{ccccccc} A & \xrightarrow{g'} & Z_j & \longrightarrow & Z_{j+1} & \longrightarrow & Z \\ {\scriptstyle k}\downarrow & & {\scriptstyle p_j}\downarrow & \swarrow & & \swarrow{\scriptstyle p_\infty} & \\ B & \xrightarrow[h]{} & Y & & & & \end{array}$$

Since the square on the left is in S_j, by construction there is a map $l\colon B \to Z_{j+1}$. Composing l with $Z_{j+1} \to Z$ provides a lift in the original square above.
$\square$

Remark: This is used to verify the factorization axiom, and to set-up inductive proofs for cofibrant objects.

Definition: A model category $\mathcal{C}$ is **cofibrantly generated** if there are sets I and J such that:
(1) The domains of I are small w.r.t. I-cell.
(2) The domains of J are small w.r.t. J-cell.
(3) The fibrations are the maps with the RLP w.r.t. J.
(3) The acyclic fibrations are the maps with the RLP w.r.t. I.

Proposition: If $\mathcal{C}$ is cofibrantly generated with generating sets I and J, then:
(1) The cofibrations are the maps in I-cof.
(2) The acyclic cofibrations are maps in J-cof.

Example: For $\mathcal{C} = s\mathcal{S}et$,
$I = \{\partial\Delta[n] \to \Delta[n]\}_n$ and $J = \{\Delta^k[n] \to \Delta[n]\}_{k,n}$.

Example: As above, $\mathcal{C} = \mathcal{C}h_R^{proj}$ and $\mathcal{C} = \mathcal{T}op^{w.e.}$.

Recognition Theorem: Let $\mathcal{C}$ be a category that is closed under all limits and colimits and let W be a subcategory of $\mathcal{C}$ that is closed under retracts and satisfies the "two out of three" axiom. If I and J are sets of maps in $\mathcal{C}$, then $\mathcal{C}$ has a cofibrantly generated model structure determined by W, I and J if and only if the following are satisfied:
(1) The domains of I are small w.r.t. I-cell;
(2) The domains of J are small w.r.t. J-cell;
(3)J-cofibrations are I-cofibrations and in W;
(4)Every map with the RLP w.r.t. I is in W and has the RLP w.r.t. J; and
(5)Either (a) any I-cofibration in W is a J-cofibration, or (b)any map in W with the RLP w.r.t. J has the RLP w.r.t. I.

A map is **K-injective** if it has the RLP w.r.t. K.

Define *fibrations* to be the J-injectives.
Define *cofibrations* to be I-cofibrations.

Proof:

(3) and (5a) show
J-cof $=$ I-cof $\cap W=$ acyclic cofibrations.

(4) and (5b) show
I-inj $=$ $W\cap$ J-inj $=$ acyclic fibrations.

(1) and (2) give factorizations into
I-cofibration and I-injective and
J-cofibration and J-injective.

I-cofibrations have LLP w.r.t. I-injectives.

J-cofibrations have LLP w.r.t. J-injectives.

Localizations: One can use this theorem and the Bousfield-Smith cardinality argument to verify that certain localization model structures exist. (left proper, combinatorial; left proper cellular)

Left localization enlarges W, cofibrations stay the same, so fibrations decrease. Local objects are the new fibrant objects.

For example, $s\mathcal{S}et_{h_*}$ (or $s\mathcal{S}et_{\mathbb{Q}}$) is a cofibrantly generated model category with W the h_*-equivalences (maps which induce isomorphisms in h_* (or $H_*(,\mathbb{Q})$)). Here $I = I_{s\mathcal{S}et} = \{\partial\Delta[n] \to \Delta[n]\}$. Then J is a set of representatives of the isomorphism classes of monomorphisms $f\colon A \to B$ that are h_*-equivalences where A and B are of "size" less than some fixed cardinal γ.

Motivic homotopy theory: ($\mathbb{A}^1$-homotopy theory) Morel and Voevodsky start with a model category on simplicial sheaves where the weak equivalences are the maps which induce weak equivalences on all stalks. Then they localize with respect to maps $X \times \mathbb{A}^1 \to X$.

Lifting model structures:
Assume given an adjoint pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ with F the left adjoint and $\mathcal{C}$ a cofibrantly generated model category. Make the following definitions:
f in $\mathcal{D}$ is a *fibration* iff $G(f)$ is a fibration in $\mathcal{C}$.
f in $\mathcal{D}$ is a *weak equivalence* iff $G(f)$ is so in $\mathcal{C}$.
f in $\mathcal{D}$ is a *cofibration* iff it has the LLP w.r.t. the acyclic fibrations.

Lifting Lemma:
[Crans; Schwede-Shipley; Berger-Moerdijk]
This defines a **lifted model structure** on $\mathcal{D}$ if
(1)F preserves small objects and
(2)any map in $F(J_{\mathcal{C}})$-cell is a weak equivalence in $\mathcal{D}$.

Moreover, $\mathcal{D}$ is *cofibrantly generated* with $I_{\mathcal{D}} = F(I_{\mathcal{C}})$ and $J_{\mathcal{D}} = F(J_{\mathcal{C}})$.

Also, F and G are Quillen functors.

(1) holds if G preserves filtered colimits.

Applications: rings, algebras and modules over symmetric spectra or gamma spaces

Quillen's path object argument:
[Quillen, II.4; Schwede-Shipley, A.3; Rezk 7.6]
Recall: a *path object* for X is a factorization of the diagonal $X \xrightarrow{\sim} X^I \twoheadrightarrow X \times X$.

If (a) $\mathcal{D}$ has a fibrant replacement functor and
(b) $\mathcal{D}$ has functorial path objects for fibrant objects, then condition (2) above holds ($F(J_{\mathcal{C}})$-cell $\subset W$).

If all objects are fibrant in $\mathcal{D}$, then (a) automatically holds.

Applications: simplicial algebras, differential graded algebras, dg-modules, operads, algebras over operads

Differential graded algebras:
Lifted model structure on DGA

$$T\colon \mathcal{C}h_R^{proj} \rightleftarrows DGA\colon U$$

(1) $T(C) = R \oplus C \oplus (C \otimes C) \oplus \cdots \oplus C^{\otimes n} \cdots$
(2) U preserves filtered colimits.
(3) All objects are fibrant.
(4) A path object for A in DGA is given by $\mathrm{Hom}_{\mathcal{C}h_R}(I, A)$.

$I = R\langle \iota_1, [0]_0, [1]_0 \rangle$ with $\partial\iota = [1] - [0]$.
I is a coassociative, counital coalgebra.
$(\Delta[0] = [0]\otimes[0], \Delta[1] = [1]\otimes[1], \Delta\iota = [0]\otimes\iota + \iota\otimes[1].)$
Counit $\eta\colon\ I \xrightarrow{\sim} R$ and two inclusions $i_0, i_1\colon\ R \to I$ induce maps $A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}h_R}(I, A) \twoheadrightarrow A \times A$.

Also DGAs over fixed commutative DGA C,
DG-modules over a fixed DGA A

Not commutative DGAs. (I is not cocommutative.)
Let $S\colon\ \mathcal{C}h_R^{proj} \to CDGA$ be the free symmetric algebra functor. If char $R \neq 0$, then $S(0 \to D^n)$ is not a quasi-isomorphism, so no such lifted model category exists.

Topic VI: Simplicial model categories

Simplicial Categories:

Definition: The **ordinal number category** Δ has objects $[n] = \{0, 1, \ldots, n\}$ and morphisms the weakly order preserving maps $\phi\colon [n] \to [m]$. Any map ϕ is a composition of maps $d^i\colon [n-1] \to [n]$ which skips i and $s^j\colon [n+1] \to [n]$ which doubles up j.

Definition: The category of **simplicial objects** in $\mathcal{C}$, $s\mathcal{C} = \mathcal{C}^{\Delta^{op}}$ is the category of contravariant functors from Δ to $\mathcal{C}$.

(1) Action (or Tensor):

$$s\mathcal{S}et \times s\mathcal{C} \to s\mathcal{C}$$
$$(K, X) \to K \otimes X$$

Simplicial sets **acts** on $s\mathcal{C}$:
$K \in s\mathcal{S}et$, $X \in s\mathcal{C}$, then $(K \otimes X)_n = \coprod_{K_n} X_n$.
Associative: $(K \times L) \otimes X \cong K \otimes (L \otimes X)$
Unital: $\Delta[0] \otimes X \cong X$

Example: If $s\mathcal{C} = s\mathcal{S}et$, $(K \otimes X)_n = K_n \times X_n$.

(2) Cotensor:

$$s\mathcal{C} \times s\mathcal{S}et^{op} \to s\mathcal{C}$$
$$(Y, K) \to Y^K$$

$K \in s\mathcal{S}et$, $Y \in s\mathcal{C}$, then Y^K is determined by
$$s\mathcal{C}(K \otimes X, Y) \cong s\mathcal{C}(X, Y^K).$$

(3) Enrichment:

$$s\mathcal{C} \times s\mathcal{C}^{op} \to s\mathcal{S}et$$
$$(X, Y) \to \mathrm{map}_{s\mathcal{C}}(X, Y)$$

For $X, Y \in s\mathcal{C}$, define $\mathrm{map}_{s\mathcal{C}}(X, Y) \in s\mathcal{S}et$ by $\mathrm{map}_{s\mathcal{C}}(X, Y)_n = s\mathcal{C}(\Delta[n] \otimes X, Y)$.

Associative composition:
$\mathrm{map}_{s\mathcal{C}}(Y, Z) \times \mathrm{map}_{s\mathcal{C}}(X, Y) \to \mathrm{map}_{s\mathcal{C}}(X, Z)$

Since $\Delta[0] \otimes X \cong X$, $\mathrm{map}_{s\mathcal{C}}(X, Y)_0 \cong s\mathcal{C}(X, Y)$.

(4) Adjoint isomorphisms:

$$\begin{aligned}\mathrm{map}_{s\mathcal{C}}(K \otimes X, Y) &\cong \mathrm{map}_{s\mathcal{C}}(X, Y^K)\\ &\cong \mathrm{map}_{s\mathcal{S}et}(K, \mathrm{map}_{s\mathcal{C}}(X, Y)).\end{aligned}$$

(1) - (4) together give $s\mathcal{C}$ the structure of a **simplicial category** (compatible tensor, cotensor and enrichment).

Examples: If $s\mathcal{C} = s\mathcal{S}et$, $(K \otimes X)_n = K_n \times X_n$ and $Y^K = \mathrm{map}_{s\mathcal{S}et}(K, Y)$.

If $s\mathcal{C} = s\mathcal{M}od_R$, $(K \otimes X)_n = R[K_n] \otimes_R X_n$ and $(Y^K) = \mathrm{map}_{s\mathcal{S}et}(K, Y)$.

$\mathcal{T}op$ is also a simplicial category: define $K \otimes X = |K| \times X$ and $Y^K = Y^{|K|}$ the topological mapping space. Then

$$\mathrm{map}_{\mathcal{T}op}(X, Y)_n = Y^{\Delta[n] \otimes X}$$

Model categories for simplicial categories
If $\mathcal{C}$ has an underlying set functor U, then we define the following.

$$L\colon s\mathcal{S}et \rightleftarrows s\mathcal{C}\colon U$$

f in $s\mathcal{C}$ is a *fibration* iff $U(f)$ is a fibration in $s\mathcal{S}et$.
f in $s\mathcal{C}$ is a *weak equivalence* iff $U(f)$ is so in $s\mathcal{S}et$.
f in $s\mathcal{C}$ is a *cofibration* iff it has the LLP w.r.t. the acyclic fibrations.

Proposition: This forms a cofibrantly generated model category for $s\mathcal{C}$ when $s\mathcal{C}$ is the category of:
(1) simplicial associative algebras
(2) simplicial Lie algebras
(3) simplicial groups
(4) simplicial commutative algebras
(5) simplicial R-modules

Proof: All objects are fibrant in each of these examples. Also, $X^{\Delta[1]}$ forms a functorial path object. This is because of the interaction between the model structures and the simplicial structures.

$$X^{\Delta[0]} = X \xrightarrow{\sim} X^{\Delta[1]} \twoheadrightarrow X \times X = X^{\Delta[0]\coprod\Delta[0]}$$

$p\colon \Delta[1] \xrightarrow{\sim} \Delta[0]$ (between cofibrant objects)
and $i\colon \Delta[0]\coprod\Delta[0] = \partial\Delta[1] \rightarrowtail \Delta[1]$

Simplicial Model Categories:
A model category $\mathcal{C}$ which is also a simplicial category is a **simplicial model category** if the following equivalent axioms hold.

SM7 Axiom: Suppose $j\colon A \rightarrowtail B$ is a cofibration and $q\colon X \twoheadrightarrow Y$ is a fibration in $\mathcal{C}$, then

$$\mathrm{map}_{\mathcal{C}}(B, X) \twoheadrightarrow \mathrm{map}_{\mathcal{C}}(A, X) \times_{\mathrm{map}_{\mathcal{C}}(A,Y)} \mathrm{map}_{\mathcal{C}}(B, Y)$$

is a fibration of simplicial sets which is acyclic if either j or q is.

Pushout product axiom: SM7 holds if and only if for any cofibration $i\colon K \rightarrowtail L$ in $s\mathcal{S}et$ and any cofibration $j\colon A \rightarrowtail B$ in $\mathcal{C}$ the map

$$i \,\Box\, j\colon (A \otimes L) \cup_{(A\otimes K)} (B \otimes K) \rightarrowtail B \otimes L$$

is a cofibration which is acyclic if either i or j is.

Third equivalent axiom: SM7 holds if and only if for any cofibration $i\colon K \rightarrowtail L$ in $s\mathcal{S}et$ and any fibration $q\colon X \twoheadrightarrow Y$ in $\mathcal{C}$ the map

$$X^L \twoheadrightarrow (X^K) \times_{(Y^K)} (Y^L)$$

is a fibration which is acyclic if either i or q is.

All of the above listed simplicial categories are simiplicial model categories.

Corollary: Let $\mathcal{C}$ be a simplicial model category. If X in $\mathcal{C}$ is fibrant, then $X^{\Delta[1]}$ is a natural path object. If X in $\mathcal{C}$ is cofibrant, then $X \otimes \Delta[1]$ is a natural cylinder object.

$$X \coprod X = \partial\Delta[1] \otimes X \rightarrowtail \Delta[1] \otimes X \xrightarrow{\sim} \Delta[0] \otimes X \cong X.$$

Corollary: If X is cofibrant and Y is fibrant in a simplicial model category $\mathcal{C}$, then

$$\pi_0 \mathrm{map}_{\mathcal{C}}(X, Y) = \mathcal{H}o(\mathcal{C})(X, Y) = [X, Y]_{\mathcal{C}}$$

Simplicial vs. Differential:

$s\mathcal{C}$ replaces differential graded objects in $\mathcal{C}$ as a place to do homological algebra.

Theorem: (Dold-Kan equivalence)

$$N\colon\ s\mathcal{M}od_R \rightleftarrows \mathcal{C}h_R\colon\ \Gamma$$

(N, Γ) is an equivalence of categories as well as a Quillen equivalence of model categories.

For $X \in s\mathcal{M}od_R$, the **normalized** chain complex NX is given by moding out by the degneracies:

$$(NX)_n = X_n / s_0 X_{n-1} + \cdots + s_{n-1} X_{n-1}$$

with

$$\partial_n = \Sigma_0^n (-1)^n d_i.$$

Then the inverse functor Γ just adds degeneracies back in. For example,

$$\Gamma(C.)_2 = C_2 \oplus s_0 C_1 \oplus s_1 C_1 \oplus s_0 s_0 C_0.$$

There is another chain complex $N'X$:

$$(N'X)_n = \cap_i^n \ker d_i\colon\ X_n \to X_{n-1}$$

with differential d_0 such that $N'X_n \to X_n \to NX_n$ is an isomorphism (which takes d_0 to ∂_n above).

One can check (since X is fibrant):

$$\begin{aligned}\pi_n X &= [S^n, |X|]_{\mathcal{T}op^{w.e.}} \\ &\cong \pi_0 \mathrm{map}_{s\mathcal{S}et}(\Delta[n]/\partial\Delta[n], X) \\ &\cong H_n(N'X) \cong H_n(NX)\end{aligned}$$

Thus, N takes weak equivalences in $s\mathcal{M}od_R$ to quasi-isomorphisms in $\mathcal{C}h_R$.

$N\colon\ s\mathcal{R}ing \to DGA$ is a Quillen equivalence (but not an equivalence of categories).

For simplicial *commutative* rings,
$N\colon\ sComm \to CDGA$ is not a Quillen equivalence (except in characteristic zero).

Quillen developed a notion of homology for any model category (given by derived functors of abelianization).

For $sComm$ this produces **André-Quillen cohomology** which is an important invariant.

Approximating simplicial model categories:

The **Dwyer-Kan hammock localization ('80)** defines for any model category $\mathcal{C}$ a category $L(C, W)$ enriched over simplicial sets such that

$$\pi_0 \mathrm{map}_{L(C,W)}(X, Y) = \mathcal{H}o(\mathcal{C})(X, Y) = [X, Y]_{\mathcal{C}}$$

If $\mathcal{C}$ is a simplicial model category, then

$$\mathrm{map}_{L(C,W)}(X, Y) \sim \mathrm{map}_{\mathcal{C}}(X, Y).$$

More recently, for model categories with functorial factorizations (e.g., cofibrantly generated model categories), one can use "framings" to construct tensors, cotensors and enrichments with nice properties. (See Hovey.)

Theorem: [Dugger '01] If $\mathcal{C}$ is either left proper and combinatorial or left proper and cellular, then $\mathcal{C}$ is Quillen equivalent to a simplicial model category structure on $s\mathcal{C}$.

Basic summary: One can usually assume one has a simplicial model category.

Topic VII: Monoidal model categories

Definition: A **monoidal model category** is a model category $\mathcal{C}$ with a monoidal product $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ and unit $\mathbb{I}$ such that:

(1) **Pushout product axiom:**
If $i\colon A \to B$ and $j\colon X \to Y$ are cofibrations in $\mathcal{C}$ then

$$i \,\square\, j\colon (A \otimes Y) \cup_{(A\otimes X)} (B \otimes X) \rightarrowtail B \otimes Y$$

is a cofibration which is acyclic if either i or j is.

(2) Let $q\colon c\mathbb{I} \xrightarrow{\sim} \mathbb{I}$ be a cofibrant replacement for the unit $\mathbb{I}$. If X is cofibrant, then $q \otimes id\colon c\mathbb{I} \otimes X \xrightarrow{\sim} \mathbb{I} \otimes X = X$ and $id \otimes q\colon X \otimes c\mathbb{I} \xrightarrow{\sim} X \otimes \mathbb{I} = X$ are weak equivalences.

Examples:
(1) $s\mathcal{S}et$ is a monoidal model category.
(2) $\mathcal{C}h_R^{proj}$ is a monoidal model category (R comm.).
($\mathcal{C}h_R^{inj}$ is not an example.)
(3) Sp^{Σ} is a monoidal model category.
(4) $\mathcal{M}_S$ is a monoidal model category.

Proposition: If $\mathcal{C}$ is a monoidal model category, then $\mathcal{H}o(\mathcal{C})$ is a monoidal category under the left derived product $\otimes^L$.

Lemma: Let $\mathcal{C}$ be cofibrantly generated with generating sets I and J. If $i \,\square\, i'$ is a cofibration for any $i, i' \in I$ and if $i \,\square\, j$ is an acyclic cofibration for any $i \in I$ and $j \in J$, then the pushout product axiom holds in general.

Definition: Let $\mathcal{C}$ be a monodial model category. A $\mathcal{C}$-model category is a model category $\mathcal{D}$ with an action by $\mathcal{C}$, $\mathcal{D} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{D}$, such that
(1) If $i\colon A \to B$ is a cofibration in $\mathcal{D}$ and $j\colon X \to Y$ is a cofibration in $\mathcal{C}$ then

$$i \,\square\, j\colon (A \otimes Y) \cup_{(A \otimes X)} (B \otimes X) \rightarrowtail B \otimes Y$$

is a cofibration in $\mathcal{D}$ which is acyclic if either i or j is.
(2) Let $q\colon c\mathbb{I} \xrightarrow{\sim} \mathbb{I}$ be a cofibrant replacement for the unit $\mathbb{I}$ in $\mathcal{C}$. If X is cofibrant, then $id \otimes q\colon X \otimes c\mathbb{I} \xrightarrow{\sim} X \otimes \mathbb{I} = X$ is a weak equivalence.

Examples:
(1) $s\mathcal{S}et$-model categories = simplicial model categories.
(2) DG-Mod$_A$ is a $\mathcal{C}h_R^{proj}$-model category for any DGA A over R.
(3) $\mathcal{C}h_R^{inj}$ is a $\mathcal{C}h_R^{proj}$-model category.
(4) $\mathcal{M}od_R$ is a Sp^Σ-model category (or $\mathcal{M}_S$-model category) for R a ring spectrum in Sp^Σ (or $\mathcal{M}_S$).

One can define rings, modules and algebras for any monoidal category $\mathcal{C}$. (For example: $\mathcal{C}h_R$, Sp^Σ, $\mathcal{M}_S$)

Quillen's path-object argument doesn't apply for lifting model categories over Sp^Σ.

Sp^Σ: no monoidal fibrant replacement functor
Functorial path objects do exist. (weak equivalences and fibrations are lifted from Sp^Σ.)

Recall the lifting lemma: $F\colon\ \mathcal{C} \rightleftarrows \mathcal{D} \colon\ G$ ..... if any map in $F(J_\mathcal{C})$-cell is a weak equivalence in $\mathcal{D}$.

Analyze pushouts over $F(J_\mathcal{C})$ in categories of modules and rings.

Let $J \wedge \mathcal{C}$ denote the class of maps $A \wedge Z \to B \wedge Z$ with $A \to B$ in J and Z any object in $\mathcal{C}$.

Monoid axiom: A cofibrantly generated, monoidal model category satisfies the *monoid axiom* if any map in $(J_\mathcal{C} \wedge \mathcal{C})$-cell is a weak equivalence.

Theorem: [Schwede-Shipley '00]
Let $\mathcal{C}$ be a cofibrantly generated, monoidal model category which satisfies the monoid axiom. (Assume all objects in $\mathcal{C}$ are small.) Then there are cofibrantly generated lifted model structures for R-modules (R a monoid) and for R-algebras (R a commuatative monoid).

Moreover, if $\mathbb{I}_{\mathcal{C}}$ is cofibrant, then every cofibrant R-algebra is also cofibrant as an R-module.

Theorem: [SS '00]
Assume $\mathbb{I}_{\mathcal{C}}$ is cofibrant and $- \wedge_R N$ preserves weak equivalences for any cofibrant left R-module N.

(1)If $f : R \xrightarrow{\sim} S$ is a weak equivalence of monoids, then $-\wedge_R S\colon\ \mathcal{M}od_R \rightleftarrows \mathcal{M}od_S\colon\ U$ is a Quillen equivalence.
(2)If $f : R \xrightarrow{\sim} S$ is a weak equivalence of commutative monoids, then $- \wedge_R S\colon\ \mathcal{A}lg_R \rightleftarrows \mathcal{A}lg_S\colon\ U$ is a Quillen equivalence.

Verifying the monoid axiom:

To show that the monoid axiom holds in $\mathcal{C}h_R^{proj}$ we use the fact that $\mathcal{C}h_R^{inj}$ is a $\mathcal{C}h_R^{proj}$-model category. (See proof below.)

The argument for Sp^Σ is similar, one uses the action of a projective model structure on an injective model structure.

Proof: In $\mathcal{C}h_R^{inj}$ all objects are cofibrant. So, the pushout product axiom implies that $(A \overset{\sim}{\rightarrowtail} B) \,\square\, (0 \rightarrowtail Z) = A \wedge Z \overset{\sim}{\rightarrowtail} B \wedge Z$ is an acylcic cofibration in $\mathcal{C}h_R^{inj}$ whenever $A \overset{\sim}{\rightarrowtail} B$ is an acyclic cofibration in $\mathcal{C}h_R^{proj}$. If $J \wedge \mathcal{C}$ consists of acyclic cofibrations (in $\mathcal{C}h_R^{inj}$), then pushouts and colimits will also be acyclic cofibrations (in $\mathcal{C}h_R^{inj}$) and hence also weak equivalences.

Equivalences of monoidal model categories: [SS '03] If $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent monoidal model categories with lifted model structures on their categories of rings (monoids), when are $\mathcal{R}ing_{\mathcal{C}}$ and $\mathcal{R}ing_{\mathcal{D}}$ Quillen equivalent?

Definitions: $R : (\mathcal{D}, \otimes) \to (\mathcal{C}, \wedge)$ is **lax monoidal** if there is a map $\nu\colon \mathbb{I}_{\mathcal{D}} \to R(\mathbb{I}_{\mathcal{C}})$ and natural associative and unital maps

$$\phi\colon RX \wedge RY \to R(X \otimes Y).$$

If R has a left adjoint L then the lax monoidal structure on R induces a **lax comonoidal** (*op-lax monoidal*) structure on L. Namely, there is a map $\widetilde{\nu}\colon L\mathbb{I}_{\mathcal{D}} \to \mathbb{I}_{\mathcal{C}}$ and natural associative and unital maps:

$$\widetilde{\phi}\colon L(A \wedge B) \to LA \otimes LB$$

L is **strong monoidal** if $\widetilde{\phi}$ and $\widetilde{\nu}$ are isomorphisms. If L is strong monoidal, then R is lax monoidal.

Definitions: Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be Quillen functors between monoidal model categories. If L is strong monoidal and $\mathbb{I}_{\mathcal{D}}$ is cofibrant, L and R are called **strong monoidal Quillen functors**.

L and R are **weak monoidal Quillen functors** if R is lax monoidal and the following two conditions hold:
(1)$\widetilde{\phi}\colon L(A \wedge B) \xrightarrow{\sim} LA \otimes LB$ is a weak equivalence in $\mathcal{C}$ whenever A and B are cofibrant in $\mathcal{D}$ and
(2) if $c\mathbb{I}_{\mathcal{D}} \xrightarrow{\sim} \mathbb{I}_{\mathcal{D}}$ is a cofibrant replacement, then $L(c\mathbb{I}_{\mathcal{D}}) \to L(\mathbb{I}_{\mathcal{D}}) \xrightarrow{\widetilde{\nu}} \mathbb{I}_{\mathcal{C}}$ is a weak equivalence in $\mathcal{C}$.

Assume $\mathcal{C}$ and $\mathcal{D}$ are monoidal model categories such that there are lifted model structures on $\mathcal{R}ing_{\mathcal{C}}$ and $\mathcal{R}ing_{\mathcal{D}}$.

Theorem:[SS '03] If $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ is a weak (or strong) monoidal Quillen equivalence and $\mathbb{I}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{D}}$ are cofibrant, then $L'\colon \mathcal{R}ing_{\mathcal{C}} \rightleftarrows \mathcal{R}ing_{\mathcal{D}}\colon R$ is a Quillen equivalence on the lifted model structures.

One also gets Quillen equivalences between respective categories of modules over $\mathcal{C}$ and over $\mathcal{D}$ (assuming the lifted model structures exist).

Proof when L is strong monoidal (so $L = L'$):
Since the model categories are lifted, weak equivalences and fibrations are determined on the underlying categories. Thus, R preserves weak equivalences and fibrations and is a right Quillen functor.

To check that L and R form a Quillen equivalence, let A be cofibrant in $\mathcal{R}ing_{\mathcal{C}}$ and B be fibrant in $\mathcal{R}ing_{\mathcal{D}}$ (and thus also fibrant in $\mathcal{D}$). We need to show that

$$LA \xrightarrow{\sim} B \text{ if and only if } A \xrightarrow{\sim} RB.$$

Since $\mathbb{I}_{\mathcal{C}}$ is cofibrant, A is also cofibrant in $\mathcal{C}$. Since $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent, this follows. $\square$

For a weak monoidal Quillen pair, one shows that $LA \xrightarrow{\sim} L'A$ is a weak equivalence for A cofibrant in $\mathcal{R}ing_{\mathcal{C}}$.

Strong monoidal examples: There are strong monoidal Quillen equivalences connecting all of the new monoidal model categories of spectra: (simplicial and topological) symmetric spectra [HSS, MMSS], orthogonal spectra [MMSS], simplicial functors [Lyd98] and W-spaces [MMSS] S-modules [EKMM].

Weak monoidal example:

$$N\colon\ s\mathcal{M}od_R \rightleftarrows \mathcal{C}h_R\colon\ \Gamma$$

(N, Γ) is a weak monoidal Quillen equivalence which is not strong. (N, Γ) are not monoidally adjoint ($id \to \Gamma N$ is not monoidal).

Corollary: There is a Quillen equivalence between the categories of connective differential graded k-algebras and simplicial k-algebras.

$$DGA_k \rightleftarrows s(\mathcal{A}lg_k)$$

Stable extension:

$$\mathcal{C}H \rightleftarrows Sp^{\Sigma}(\mathcal{C}h) \rightleftarrows Sp^{\Sigma}(s(\mathcal{A}b)) \rightleftarrows H\mathbb{Z}\text{-Mod}$$

a string of weak monoidal Quillen equivalences which induces a Quillen equivalence

$$DGA \rightleftarrows H\mathbb{Z}\text{-Alg}$$

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