Topological equivalences of differential graded algebras

(Joint work with D. Dugger)

"Abelian groups up to homotopy"
spectra $\Longleftrightarrow$ generalized cohomology theories

Examples:

1. **Ordinary cohomology:**

For A any abelian group, $H^*(X;A) = [X_+, K(A,n)]$.

Eilenberg-Mac Lane spectrum, denoted HA.
$HA_n = K(A,n)$ for $n \geq 0$.

The coefficients of the theory are given by

$$HA^*(\mathrm{pt}) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

2. **Hypercohomology:**

For $C.$ any chain complex of abelian groups,
$\mathbb{H}^s(X; C.) \cong \oplus_{q-p=s} H^p(X; H_q(C.))$.
Just a direct sum of shifted ordinary cohomologies.

$HC.^*(\text{pt}) = H_*(C.)$.

3. **Complex K-theory:**

$K^*(X)$; associated spectrum denoted K.

$$K_n = \begin{cases} U & n = \text{odd} \\ BU \times \mathbb{Z} & n = \text{even} \end{cases}$$

$$K^*(\text{pt}) = \begin{cases} 0 & * = \text{odd} \\ \mathbb{Z} & * = \text{even} \end{cases}$$

4. **Stable cohomotopy:**

$\pi_S^*(X)$; associated spectrum denoted $\mathbb{S}$.

$\mathbb{S}_n = S^n$, $\mathbb{S}$ is the *sphere spectrum.*

$\pi_S^*(\text{pt}) = \pi_{-*}^S(\text{pt}) =$ stable homotopy groups of spheres. These are only known in a range.

“Rings up to homotopy”
ring spectra $\Longleftrightarrow$gen. coh. theories with a product

1. For R a ring, HR **is a ring spectrum.**
The cup product gives a graded product:
$HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$

Induced by $K(R,p) \wedge K(R,q) \to K(R,p+q)$.

Definition. $X \wedge Y = X \times Y/(X \times pt) \cup (pt \times Y)$.

2. For $A.$ a differential graded algebra (DGA), $HA.$ **is a ring spectrum.** Product induced by $\mu : A. \otimes A. \to A.$, or $A_p \otimes A_q \to A_{p+q}$.

The groups $\mathbb{H}(X; A.)$ are still determined by $H_*(A)$, but the product structure is *not* determined $H_*(A)$.

3. K **is a ring spectrum;**
Product induced by tensor product of vector bundles.

4. $\mathbb{S}$ **is a commutative ring spectrum.**

Definitions

"Definition." A *ring spectrum* is a sequence of pointed spaces $R = (R_0, R_1, \cdots, R_n, \cdots)$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

Definition. The *suspension* of a based space X is $\Sigma X = S^1 \wedge X \cong (CX \cup_X CX)/\tilde{\ }\mathrm{pt}$.

(Here I drew a representation of the suspension of X.)

Definition. A *spectrum* F is a sequence of pointed spaces $(F_0, F_1, \cdots, F_n, \cdots)$ with structure maps $\Sigma F_n \to F_{n+1}$.

Example: $\mathbb{S}$ a commutative ring spectrum

Structure maps: $\Sigma S^n = S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$.

Product maps: $S^p \wedge S^q \xrightarrow{\cong} S^{p+q}$.

Actually, must be more careful here. For example: $S^1 \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge S^1$ is a degree -1 map.

History of spectra and $\wedge$

Boardman in 1965 defined spectra and $\wedge$*.* $\wedge$ is only commutative and associative up to homotopy.

A_∞ *ring spectrum* = best approximation to associative ring spectrum.

E_∞ *ring spectrum* = best approximation to commutative ring spectrum.

Lewis in 1991: No good $\wedge$ *exists.*
Five reasonable axioms $\Longrightarrow$ no such $\wedge$.

Since 1997, lots of good categories of spectra exist! (with $\wedge$ that is commutative and associative.)
1. 1997: Elmendorf, Kriz, Mandell, May
2. 2000: Hovey, S., Smith
3, 4 and 5 ... Lydakis, Schwede, ...

Theorem.(Mandell, May, Schwede, S. 2001)
All above models define the same homotopy theory.

Spectral Algebra

Given the good categories of spectra with $\wedge$, one can easily do algebra with spectra.

Definitions:

A *ring spectrum* is a spectrum R with an associative and unital multiplication $\mu : R \wedge R \longrightarrow R$ (with unit $\mathbb{S} \to R$).

An *R-module spectrum* is a spectrum M with an associative and unital action $\alpha : R \wedge M \longrightarrow M$.

$\mathbb{S}$-*modules* are spectra.
$S^1 \wedge F_n \longrightarrow F_{n+1}$ iterated gives $S^p \wedge F_q \longrightarrow F_{p+q}$.
Fits together to give $\mathbb{S} \wedge F \longrightarrow F$.

$\mathbb{S}$-*algebras* are ring spectra.

Homological Algebra vs. Spectral Algebra

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	$\mathbb{S}$
$\mathbb{Z}$-Mod $= \mathcal{A}b$	d.g.-Mod $= \mathcal{C}h$	$\mathbb{S}$-Mod $= \mathcal{S}pectra$
$\mathbb{Z}$-Alg $=$ $\mathcal{R}ings$	d.g.-Alg $=$ $\mathcal{D}GAs$	$\mathbb{S}$-Alg $=$ $\mathcal{R}ing\ spectra$

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	H$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-Mod	d.g.-Mod	H$\mathbb{Z}$-Mod	$\mathbb{S}$-Mod
$\mathbb{Z}$-Alg	d.g.-Alg	H $\mathbb{Z}$-Alg	$\mathbb{S}$-Alg

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	H$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-Mod	d.g.-Mod	H$\mathbb{Z}$-Mod	$\mathbb{S}$-Mod
$\mathbb{Z}$-Alg	d.g.-Alg	H $\mathbb{Z}$-Alg	$\mathbb{S}$-Alg
$\cong$	quasi-iso	weak equiv.	weak equiv.

Quasi-isomorphisms are maps which induce isomorphisms in homology.

Weak equivalences are maps which induce isomorphisms on the coefficients.

$\mathbb{Z}$	$\mathbb{Z}$ (d.g.)	H$\mathbb{Z}$	$\mathbb{S}$
$\mathbb{Z}$-Mod	d.g.-Mod	H$\mathbb{Z}$-Mod	$\mathbb{S}$-Mod
$\mathbb{Z}$-Alg	d.g.-Alg	H $\mathbb{Z}$-Alg	$\mathbb{S}$-Alg
$\cong$	quasi-iso	weak equiv.	weak equiv.
	$\mathcal{D}(\mathbb{Z}) = \mathcal{C}h[\text{q-iso}]^{-1}$	$\mathcal{H}o(H\mathbb{Z}\text{-Mod})$	$\mathcal{H}o(\mathcal{S}) = \mathcal{S}pectra[\text{wk.eq.}]^{-1}$

Theorem. Columns two and three are equivalent up to homotopy.

(1) (Robinson '87) $\mathcal{D}(\mathbb{Z}) \simeq_{\Delta} \mathcal{H}o(H\mathbb{Z}\text{-Mod})$.

(2) (Schwede-S.) $\mathcal{C}h \simeq_{\text{Quillen}} H\mathbb{Z}$-Mod.

(3) (S.) Associative $\mathcal{D}GA \simeq_{\text{Quillen}}$ Assoc. $H\mathbb{Z}$-Alg.

(4) (S.) For $A.$ a DGA,
d.g. $A.$-Mod $\simeq_{\text{Quillen}} HA.$-Mod
and $\mathcal{D}(A.) \simeq_{\Delta} \mathcal{H}o(HA.\text{-Mod})$.

Consider DGAs as ring spectra

(Here I drew a picture with each component representing a quasi-isomorphism type of DGAs and, in a different color, components of weak-equivalence types of ring spectra. Some components of ring spectra could contain several components of DGAs, some could contain none.)

Definition. Two DGAs $A.$ and $B.$ are *topologically equivalent* if their associated $H\mathbb{Z}$-algebras $HA.$ and $HB.$ are equivalent as ring spectra ($\mathbb{S}$-algebras).

Theorem. If $A.$ and $B.$ are topologically equivalent DGAs, then $\mathcal{D}(A.) \simeq_\Delta \mathcal{D}(B.)$.

Proof. This follows since

$$\text{d.g. } A.\text{-Mod} \simeq_Q HA.\text{-Mod} \simeq_Q HB.\text{-Mod} \simeq_Q \text{d.g. } B.\text{-Mod}$$

Equivalences of module categories

(Morita 1958) Any equivalence of categories R-Mod $\cong$ R'-Mod is given by tensoring with a bimodule.

(Rickard 1989, 1991) Any derived equivalence of rings $\mathcal{D}(R) \cong_{\Delta} \mathcal{D}(R')$ is given by tensoring with a complex of bimodules (a *tilting complex*).

(Schwede-S. 2003) Any Quillen equivalence of module spectra R-Mod $\simeq_Q$ R'-Mod is given by smashing with a bimodule spectrum (a *tilting spectrum*).

(Dugger-S.) Example below shows that for derived equivalences of DGAs one must consider tilting spectra, not just tilting complexes.

In fact, there is also an example of a derived equivalence of DGAs which doesn't come from a tilting spectrum (because it doesn't come from an underlying Quillen equivalence.) (This example is based on work by (Schlichting 2002).)

Example:

$A = \mathbb{Z}[e_1]/(e^4)$ with $de = 2$ and $A' = H_*A$

$$= \Lambda_{\mathbb{Z}/2}(\alpha_2)$$

(Here I drew representations of these two DGAs.)

A and A' are *not* quasi-isomorphic,
(although $H_*A \cong H_*A'$.)

Claim: A and A' are topologically equivalent.
Or, $HA \simeq HA'$ as ring spectra.

Use HH^* and THH^*:

For a ring R and an R-bimodule M, DGAs with non-zero homology $H_0 = R$ and $H_n = M$ are classified by $HH^{n+2}_{\mathbb{Z}}(R; M)$.

Topological Hochschild cohomology
Using $\wedge$ in place of $\otimes$ one can mimic the definition of HH for spectra to define THH.

In particular, $HH^*_{\mathbb{Z}}(R; M) = THH^*_{H\mathbb{Z}}(HR; HM)$.

Just as above, ring spectra are classified by $THH^{n+2}_{\mathbb{S}}(HR; HM)$.

A and A' are thus classified in these two settings by letting $R = \mathbb{Z}/2$, $M = \mathbb{Z}/2$ and $n = 2$.

$\mathbb{S} \to H\mathbb{Z}$ induces
$\Phi : HH^*_{\mathbb{Z}}(\mathbb{Z}/2; \mathbb{Z}/2) \to THH^*_{\mathbb{S}}(\mathbb{Z}/2; \mathbb{Z}/2)$.

One can calculate that A and A' correspond to different elements in HH^4 which get mapped to the same element in THH^4.

Compute:

$$HH^*_{\mathbb{Z}}(\mathbb{Z}/2;\mathbb{Z}/2) = \mathbb{Z}/2[\sigma_2]$$

(Franjou, Lannes, and Schwartz 1994)

$$\begin{aligned} THH^*_{\mathbb{S}}(\mathbb{Z}/2;\mathbb{Z}/2) &= \Gamma_{\mathbb{Z}/2}[\tau_2] \\ &\cong \Lambda_{\mathbb{Z}/2}(e_1, e_2, \ldots), \quad \deg(e_i) = 2^i. \end{aligned}$$

To compute $\Phi : HH^*_{\mathbb{Z}}(\mathbb{Z}/2) \to THH^*_{\mathbb{S}}(\mathbb{Z}/2)$:

In HH^2: $\sigma \leftrightarrow \mathbb{Z}/4$ and $0 \leftrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$
In THH^2: $\tau \leftrightarrow H\mathbb{Z}/4$ and $0 \leftrightarrow H(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$

So $\Phi(\sigma) = \tau$.

In HH^4: $\sigma^2 \leftrightarrow A$ and $0 \leftrightarrow A'$.

$\Phi(\sigma^2) = \Phi(0) = 0$ since $\tau^2 = 0$ and
Φ is a ring homomorphism.

So $HA \simeq HA'$ as ring spectra,
although $A \not\simeq A'$ as DGAs.

It follows that $\mathcal{D}(A) \simeq_{\Delta} \mathcal{D}(A')$.

Example: There exist two DGAs A and B such that

$$\mathcal{D}_A \cong_\Delta \mathcal{D}_B, \text{ but}$$
$$\text{d.g.}A\text{-mod} \not\simeq_Q \text{d.g.}B\text{-mod}$$

Based on Marco Schlichting's example ($p > 3$):

$$\mathcal{H}o(\text{Stmod}(\ \mathbb{Z}/p[\epsilon]/\epsilon^2)) \cong_\Delta \mathcal{H}o(\text{Stmod}(\ \mathbb{Z}/p^2)),$$
$$\text{but}$$
$$\text{Stmod}(\ \mathbb{Z}/p[\epsilon]/\epsilon^2) \not\simeq_Q \text{Stmod}(\mathbb{Z}/p^2)$$

One can find DGAs A and B such that:

$$\text{Stmod}(\ \mathbb{Z}/p[\epsilon]/\epsilon^2) \simeq_Q \text{d.g.}A\text{-mod}$$

$$\text{Stmod}(\ \mathbb{Z}/p^2) \simeq_Q \text{d.g.}B\text{-mod}$$

Here A and B are the endomorphism DGAs of the Tate resolution of a generator ($\mathbb{Z}/p$ in both cases):

$A = \mathbb{Z}/p[x_1, x_1^{-1}]$ with $d = 0$.

$B = \mathbb{Z}[x_1, x_1^{-1}]\langle e_1\rangle/e^2 = 0, ex + xe = x^2$ with $de = p$ and $dx = 0$.