Topological equivalences of differential graded algebras
(Joint work with D. Dugger)

“Abelian groups up to homotopy”
spectra \(\iff\) generalized cohomology theories

Examples:

1. **Ordinary cohomology:**

For \(A\) any abelian group, \(H^*(X; A) = [X_+, K(A, n)]\).

Eilenberg-Mac Lane spectrum, denoted \(HA\).
\(HA_n = K(A, n)\) for \(n \geq 0\).

The coefficients of the theory are given by
\[
HA^*(pt) = \begin{cases} 
A \ast = 0 \\
0 \ast \neq 0 
\end{cases}
\]
2. **Hypercohomology:**

For $C.$ any chain complex of abelian groups, 
\[ \mathbb{H}^s(X; C.) \cong \oplus_{q-p=s} H^p(X; H_q(C.)) \].

Just a direct sum of shifted ordinary cohomologies.

\[ HC.(pt) = H_*(C.) \].

3. **Complex K-theory:**

\[ K^*(X) \]; associated spectrum denoted $K$.

\[ K_n = \begin{cases} U & n = \text{odd} \\ BU \times \mathbb{Z} & n = \text{even} \end{cases} \]

\[ K^*(pt) = \begin{cases} 0 & * = \text{odd} \\ \mathbb{Z} & * = \text{even} \end{cases} \]

4. **Stable cohomotopy:**

\[ \pi^*_S(X) \]; associated spectrum denoted $S$.

\[ S_n = S^n \], $S$ is the *sphere spectrum*.

\[ \pi^*_S(pt) = \pi^*_S(pt) = \text{stable homotopy groups of spheres. These are only known in a range.} \]
“Rings up to homotopy”
ring spectra $\iff$ gen. coh. theories with a product

1. For $R$ a ring, $HR$ is a ring spectrum.
The cup product gives a graded product:
$HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$

Induced by $K(R, p) \land K(R, q) \to K(R, p + q)$.

Definition. $X \land Y = X \times Y/(X \times pt) \cup (pt \times Y)$.

2. For $A$, a differential graded algebra (DGA),
$HA$ is a ring spectrum. Product induced by
$\mu: A \otimes A \to A$, or $A_p \otimes A_q \to A_{p+q}$.

The groups $H(X; A)$ are still determined by $H_*(A)$,
but the product structure is not determined $H_*(A)$.

3. $K$ is a ring spectrum;
Product induced by tensor product of vector bundles.

4. $S$ is a commutative ring spectrum.
Definitions

“Definition.” A ring spectrum is a sequence of pointed spaces $R = (R_0, R_1, \cdots, R_n, \cdots)$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

Definition. The suspension of a based space $X$ is

$\Sigma X = S^1 \wedge X \cong (C X \cup_X C X)/\sim \text{pt.}$

(Here I drew a representation of the suspension of $X$.)

Definition. A spectrum $F$ is a sequence of pointed spaces $(F_0, F_1, \cdots, F_n, \cdots)$ with structure maps $\Sigma F_n \to F_{n+1}$.

Example: $\mathbb{S}$ a commutative ring spectrum

Structure maps: $\Sigma S^n = S^1 \wedge S^n \overset{\cong}{\to} S^{n+1}$.

Product maps: $S^p \wedge S^q \overset{\cong}{\to} S^{p+q}$.

Actually, must be more careful here. For example: $S^1 \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge S^1$ is a degree $-1$ map.
History of spectra and $\wedge$

*Boardman in 1965 defined spectra and $\wedge$. $\wedge$ is only commutative and associative up to homotopy.*

$A_\infty$ *ring spectrum* = best approximation to associative ring spectrum.

$E_\infty$ *ring spectrum* = best approximation to commutative ring spectrum.

*Lewis in 1991: No good $\wedge$ exists.*

Five reasonable axioms $\implies$ no such $\wedge$.

*Since 1997, lots of good categories of spectra exist!* (with $\wedge$ that is commutative and associative.)

1. 1997: Elmendorf, Kriz, Mandell, May
2. 2000: Hovey, S., Smith
3. 4 and 5 ... Lydakis, Schwede, ...

**Theorem.** *(Mandell, May, Schwede, S. 2001)*

All above models define the same homotopy theory.
Spectral Algebra

Given the good categories of spectra with $\wedge$, one can easily do algebra with spectra.

Definitions:

A *ring spectrum* is a spectrum $R$ with an associative and unital multiplication $\mu : R \wedge R \to R$ (with unit $S \to R$).

An *$R$-module spectrum* is a spectrum $M$ with an associative and unital action $\alpha : R \wedge M \to M$.

$S$-*modules* are spectra.
$S^1 \wedge F_n \to F_{n+1}$ iterated gives $S^p \wedge F_q \to F_{p+q}$.
Fits together to give $S \wedge F \to F$.

$S$-*algebras* are ring spectra.
Homological Algebra vs. Spectral Algebra

\[
\begin{array}{ccc}
\mathbb{Z} & \mathbb{Z} \text{ (d.g.)} & S \\
\mathbb{Z}\text{-Mod} & \text{d.g.-Mod} & S\text{-Mod} \\
= Ab & = Ch & = \text{Spectra} \\
\mathbb{Z}\text{-Alg} = \text{d.g.-Alg} = S\text{-Alg} = \\
\text{Rings} & \text{DGAs} & \text{Ring spectra} \\
\end{array}
\]

\[
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} \text{ (d.g.)} & \mathbb{H}\mathbb{Z} & S \\
\mathbb{Z}\text{-Mod} & \text{d.g.-Mod} & \mathbb{H}\mathbb{Z}\text{-Mod} & S\text{-Mod} \\
\mathbb{Z}\text{-Alg} & \text{d.g.-Alg} & \mathbb{H}\mathbb{Z}\text{-Alg} & S\text{-Alg} \\
\simeq & \text{quasi-iso} & \text{weak equiv.} & \text{weak equiv.} \\
\end{array}
\]

Quasi-isomorphisms are maps which induce isomorphisms in homology.

Weak equivalences are maps which induce isomorphisms on the coefficients.
\[
\begin{array}{c|c|c|c}
\mathbb{Z} & \mathbb{Z} \text{ (d.g.)} & \mathbb{H} \mathbb{Z} & \mathcal{S} \\
\mathbb{Z} \text{-Mod} & \text{d.g.-Mod} & \mathbb{H} \mathbb{Z} \text{-Mod} & \mathcal{S} \text{-Mod} \\
\mathbb{Z} \text{-Alg} & \text{d.g.-Alg} & \mathbb{H} \mathbb{Z} \text{-Alg} & \mathcal{S} \text{-Alg} \\
\cong & \text{quasi-iso} & \text{weak equiv.} & \text{weak equiv.} \\
\mathcal{D}(\mathbb{Z}) & \mathcal{H}\mathcal{O}(\mathbb{H} \mathbb{Z} \text{-Mod}) & \mathcal{H}\mathcal{O}(\mathcal{S}) & \text{Spectra}[\text{wk.eq.}]^{-1} \\
\mathcal{C}h[q\text{-iso}]^{-1} & & & \\
\end{array}
\]

**Theorem.** Columns two and three are equivalent up to homotopy.

1. (Robinson ‘87) \( \mathcal{D}(\mathbb{Z}) \cong_{\Delta} \mathcal{H}\mathcal{O}(\mathbb{H} \mathbb{Z} \text{-Mod}) \).

2. (Schwede-S.) \( \mathcal{C}h \cong_{\text{Quillen}} \mathbb{H} \mathbb{Z} \text{-Mod} \).

3. (S.) Associative \( \mathcal{D}GA \cong_{\text{Quillen}} \text{Assoc. } \mathbb{H} \mathbb{Z} \text{-Alg} \).

4. (S.) For \( A \) a DGA,
   \[
   \text{d.g. } A \text{-Mod} \cong_{\text{Quillen}} \mathbb{H} A \text{-Mod}
   \]
   and \( \mathcal{D}(A) \cong_{\Delta} \mathcal{H}\mathcal{O}(\mathbb{H} A \text{-Mod}) \).
Consider DGAs as ring spectra

(Here I drew a picture with each component representing a quasi-isomorphism type of DGAs and, in a different color, components of weak-equivalence types of ring spectra. Some components of ring spectra could contain several components of DGAs, some could contain none.)

**Definition.** Two DGAs \( A \) and \( B \) are *topologically equivalent* if their associated \( H \mathbb{Z} \)-algebras \( HA \) and \( HB \) are equivalent as ring spectra (\( \mathbb{S} \)-algebras).

**Theorem.** If \( A \) and \( B \) are topologically equivalent DGAs, then \( \mathcal{D}(A.) \simeq_\Delta \mathcal{D}(B.) \).

**Proof.** This follows since
\[
d.g. \ A.-\text{Mod} \cong_q \mathcal{H}A.-\text{Mod} \cong_q \mathcal{H}B.-\text{Mod} \\
\cong_q \text{d.g. } B.-\text{Mod}
\]
Equivalences of module categories

(Morita 1958) Any equivalence of categories $R$-Mod $\cong R'$-Mod is given by tensoring with a bimodule.

(Rickard 1989, 1991) Any derived equivalence of rings $\mathcal{D}(R) \cong_\Delta \mathcal{D}(R')$ is given by tensoring with a complex of bimodules (a tilting complex).

(Schwede-S. 2003) Any Quillen equivalence of module spectra $R$-Mod $\simeq_Q R'$-Mod is given by smashing with a bimodule spectrum (a tilting spectrum).

(Dugger-S.) Example below shows that for derived equivalences of DGAs one must consider tilting spectra, not just tilting complexes.

In fact, there is also an example of a derived equivalence of DGAs which doesn’t come from a tilting spectrum (because it doesn’t come from an underlying Quillen equivalence.) (This example is based on work by (Schlichting 2002).)
**Example:**

\[ A = \mathbb{Z}[e_1]/(e^4) \text{ with } de = 2 \text{ and } A' = H_* A \]
\[ = \Lambda_{\mathbb{Z}/2}(\alpha_2) \]

(Here I drew representations of these two DGAs.)

\[ A \text{ and } A' \text{ are not quasi-isomorphic,} \]
\[ (\text{although } H_* A \cong H_* A'). \]

Claim: \( A \) and \( A' \) are topologically equivalent.
Or, \( HA \simeq HA' \) as ring spectra.
Use $HH^*$ and $THH^*$:

For a ring $R$ and an $R$-bimodule $M$, DGAs with non-zero homology $H_0 = R$ and $H_n = M$ are classified by $HH^Z_{n+2}(R; M)$.

**Topological Hochschild cohomology**

Using $\wedge$ in place of $\otimes$ one can mimic the definition of $HH$ for spectra to define $THH$.

In particular, $HH^*_Z(R; M) = THH^*_{HZ}(HR; HM)$.

Just as above, ring spectra are classified by $THH^n_{S}(HR; HM)$.

$A$ and $A'$ are thus classified in these two settings by letting $R = \mathbb{Z}/2$, $M = \mathbb{Z}/2$ and $n = 2$.

$S \to H\mathbb{Z}$ induces

$\Phi : HH^*_Z(\mathbb{Z}/2; \mathbb{Z}/2) \to THH^*_S(\mathbb{Z}/2; \mathbb{Z}/2)$.

One can calculate that $A$ and $A'$ correspond to different elements in $HH^4$ which get mapped to the same element in $THH^4$. 
Compute:

$$\mathrm{HH}_\mathbb{Z}^*(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[\sigma_2]$$

(Franjou, Lannes, and Schwartz 1994)

$$\mathrm{THH}_\mathbb{S}^*(\mathbb{Z}/2; \mathbb{Z}/2) = \Gamma_{\mathbb{Z}/2}[\tau_2]$$

$$\cong \Lambda_{\mathbb{Z}/2}(e_1, e_2, \ldots), \quad \deg(e_i) = 2^i.$$

To compute $\Phi : \mathrm{HH}_\mathbb{Z}^*(\mathbb{Z}/2) \to \mathrm{THH}_\mathbb{S}^*(\mathbb{Z}/2)$:

In $\mathrm{HH}^2$: $\sigma \leftrightarrow \mathbb{Z}/4$ and $0 \leftrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$

In $\mathrm{THH}^2$: $\tau \leftrightarrow H\mathbb{Z}/4$ and $0 \leftrightarrow H(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$

So $\Phi(\sigma) = \tau$.

In $\mathrm{HH}^4$: $\sigma^2 \leftrightarrow A$ and $0 \leftrightarrow A'$.

$\Phi(\sigma^2) = \Phi(0) = 0$ since $\tau^2 = 0$ and $\Phi$ is a ring homomorphism.

So $HA \simeq HA'$ as ring spectra, although $A \not\simeq A'$ as DGAs.

It follows that $\mathcal{D}(A) \simeq_{\Delta} \mathcal{D}(A')$. 
**Example:** There exist two DGAs $A$ and $B$ such that

$$\mathcal{D}_A \cong_\Delta \mathcal{D}_B, \text{ but } \text{d.g.} A\text{-mod} \not\cong_Q \text{d.g.} B\text{-mod}$$

Based on Marco Schlichting’s example ($p > 3$):

$$\mathcal{Ho}(\text{Stmod}(\mathbb{Z}/p[\epsilon]/\epsilon^2)) \cong_\Delta \mathcal{Ho}(\text{Stmod}(\mathbb{Z}/p^2)), \text{ but } \text{Stmod}(\mathbb{Z}/p[\epsilon]/\epsilon^2) \not\cong_Q \text{Stmod}(\mathbb{Z}/p^2)$$

One can find DGAs $A$ and $B$ such that:

$$\text{Stmod}(\mathbb{Z}/p[\epsilon]/\epsilon^2) \cong_Q \text{d.g.} A\text{-mod}$$

$$\text{Stmod}(\mathbb{Z}/p^2) \cong_Q \text{d.g.} B\text{-mod}$$

Here $A$ and $B$ are the endomorphism DGAs of the Tate resolution of a generator ($\mathbb{Z}/p$ in both cases):

$A = \mathbb{Z}/p[x_1, x_1^{-1}]$ with $d = 0$.

$B = \mathbb{Z}[x_1, x_1^{-1}]\langle e_1 \rangle/e^2 = 0$, $ex + xe = x^2$ with $de = p$ and $dx = 0$. 