POSTNIKOV EXTENSIONS OF RING SPECTRA

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Abstract. We give a functorial construction of k-invariants for ring spectra, and use these to classify extensions in the Postnikov tower of a ring spectrum.

Contents

1. Introduction

This paper concerns k-invariants for ring spectra and their role in classifying Postnikov extensions. Recall that a connective ring spectrum R has a Postnikov tower

$$\cdots \to P_2R \to P_1R \to P_0R \to *$$

in the homotopy category of ring spectra. The levels come equipped with compatible maps $R \to P_nR$, and the nth level is characterized by having $\pi_i(P_nR) = 0$ for $i > n$, together with the fact that $\pi_i(R) \to \pi_i(P_nR)$ is an isomorphism for $i \leq n$. In this paper we produce k-invariants for the levels of this tower, and explain their role in the following problem: if one only knows $P_{n-1}R$ together with $\pi_n(R)$ as a $\pi_0(R)$-bimodule, what are the possibilities for P_nR? Corollary 1.4 below shows in what sense the possibilities are classified by k-invariants.

1.1. **Classical k-invariants.** To explain our results further, it's useful to briefly recall the situation for ordinary topological spaces. If X is a space, let P_nX be the nth Postnikov section of X. The k-invariant is a map $P_{n-1}X \to K(\pi_nX, n+1)$ and the homotopy fiber of this map is weakly equivalent to P_nX. So P_nX can be recovered from the k-invariant, and in fact the k-invariant only depends on P_nX.

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One is tempted to say that the possibilities for P_nX are *classified by* the possible k-invariants, but this is where some care is needed.

To clarify the situation, it's useful to set $C = P_{n-1}X$ and $M = \pi_n X$. By a Postnikov n-extension of C (of type M) we mean a space Y together with a map $Y \to C$ such that $\pi_i(Y) \to \pi_i(C)$ is an isomorphism for $i \le n-1$, $\pi_n Y \cong M$, and $\pi_i Y = 0$ for $i > n$. Note that the isomorphism $\pi_n Y \cong M$ is not part of the data. The Postnikov n-extensions form a category, in which a map from $Y \to C$ to $Y' \to C$ is a weak equivalence $Y \to Y'$ making the evident triangle commute. We'll denote this category $\mathcal{M}(C, M, n)$.

For convenience, suppose that C is simply connected (so that we don't have to worry about the $\pi_1 C$ actions). One is tempted to claim that the connected components of $\mathcal{M}(C, M, n)$ are in bijective correspondence with the set of homotopy classes $[C, K(M, n+1)]$. Unfortunately, this isn't quite the case. Note that the group $\operatorname{Aut} M$ of abelian group automorphisms acts on $K(M, n+1)$ and hence on $[C, K(M, n+1)]$. If a certain k-invariant $C \to K(M, n+1)$ is 'twisted' by an automorphism of M, it gives rise to a weakly equivalent extension of C. The correct statement, it then turns out, is that if C is simply connected there is a bijection

$$\pi_0 \mathcal{M}(C, M, n) \cong [C, K(M, n+1)]/\operatorname{Aut}(M).$$

This statement is best proven by upgrading it to a statement about the homotopy type of $\mathcal{M}(C, M, n)$ (where by the homotopy type of a category we always mean the homotopy type of its nerve). One can prove that there is a homotopy fiber sequence

$$\operatorname{Map}(C, K(M, n+1)) \to \mathcal{M}(C, M, n) \to B\operatorname{Aut}(M),$$

and the resulting long exact sequence of homotopy groups gives the identification of $\pi_0 \mathcal{M}(C, M, n)$ cited above.

For a proof of this homotopy fiber sequence (together with a version when C is not simply connected) we refer the reader to [BDG, Sections 2,3]. An important part of the proof is having a simple, functorial construction of the k-invariant, and we now describe this. If $p\colon Y \to C$ is a Postnikov n-extension of type M, let D denote the homotopy cofiber of p. One can prove by a Blakers-Massey type result that $\pi_i(D) = 0$ for $i \le n$, whereas $\pi_{n+1} D \cong \pi_n Y \cong M$. Then $P_{n+1}D$ is an Eilenberg-MacLane space $K(M, n+1)$, and our k-invariant for Y is the composite

$$C \to D \to P_{n+1}D.$$

Note that in some sense this is not 'really' a k-invariant, as one does not have a specified weak equivalence $P_{n+1}D \simeq K(M, n+1)$. One can prove that different weak equivalences differ by an element of $\operatorname{Aut}(M)$, and this shows that one has a well-defined element of the orbit space $[C, K(M, n+1)]/\operatorname{Aut}(M)$.

1.2. **Results for ring spectra.** Now we jump into the category of ring spectra, and state our main results. In this paper we always work in the category of symmetric spectra from [HSS]. So 'ring spectrum' means 'symmetric ring spectrum'.

Fix $n \ge 1$. Let C be a connective ring spectrum such that $P_{n-1}C \simeq C$, and let M be a $\pi_0 C$-bimodule. By a **Postnikov extension of C of type (M, n)** one means a ring spectrum Y together with a ring map $Y \to C$ such that

(i) $\pi_i Y = 0$ for $i > n$,
(ii) $\pi_i Y \to \pi_i C$ is an isomorphism for $i \le n-1$ and

(iii) $\pi_n Y \cong M$ as $\pi_0 Y$-bimodules (where M becomes a $\pi_0 Y$-bimodule via the isomorphism $\pi_0 Y \to \pi_0 C$).

A map of Postnikov extensions from $Y \to C$ to $Y' \to C$ is a weak equivalence of ring spectra $Y \to Y'$ making the triangle commute. Denote the resulting category by $\mathcal{M}(C + (M, n))$. We call this the **moduli space** (or **moduli category**) of Postnikov extensions of C of type (M, n).

We next identify the analogues of Eilenberg-MacLane spaces. Given a C-bimodule W, one can construct a ring spectrum $C \vee W$ whose underlying spectrum is the wedge and where the multiplication comes from the bimodule structure on W—so W squares to zero under this product. We call $C \vee W$ the **trivial square zero extension** of C by W.

Given our $\pi_0(C)$-bimodule M, there is a C-bimodule HM for which $\pi_0(HM) \cong M$ and $\pi_i(HM) = 0$ for $i \neq 0$. In fact, all such bimodules are weakly equivalent (in the category of C-bimodules). One gets resulting bimodules $\Sigma^i(HM)$ for all i, and therefore ring spectra $C \vee \Sigma^i(HM)$. Throughout the paper we will abuse notation and simplify 'HM' to just 'M'—thus, we will write $C \vee \Sigma^i M$ for $C \vee \Sigma^i HM$.

Here is our main theorem:

Theorem 1.3. *Given C and M as above, there is a homotopy fiber sequence*

$$\underline{\mathcal{R}ing_{/C}}(C, C \vee \Sigma^{n+1} M) \to \mathcal{M}(C + (M, n)) \to B\,\mathrm{Aut}(M)$$

where $\mathrm{Aut}(M)$ is the group of $\pi_0(C)$-bimodule automorphisms of M. Here $\underline{\mathcal{R}ing_{/C}(X, Y)}$ denotes the homotopy mapping space from X to Y in the catgory of ring spectra over C.

Corollary 1.4. *There is a bijection of sets*

$$\pi_0 \mathcal{M}(C + (M, n)) \cong [Ho\,(\mathcal{R}ing_{/C})(C, C \vee \Sigma^{n+1} M)]/\,\mathrm{Aut}(M).$$

In the context of these results, the main difference between ring spectra and ordinary topological spaces is that there are no absolute cohomology theories for ring spectra. When dealing with ring spectra, one always deals with *relative* cohomology theories. (For a nice explanation of this phenomenon in the commutative case, see the introduction to [BM].) Thus, in the above results one is forced to always work over C: the analogue of the Eilenberg-MacLane space is the ring spectrum $C \vee \Sigma^{n+1} M$, and the mapping spaces must be computed in the category of ring spectra *over* C. Aside from these differences, the statements for ring spectra and topological spaces are quite similar.

The above results can actually be extended, so that they apply not just to ring spectra but to algebras over a given connective, commutative ring spectrum R. This is the form in which we will actually prove them (see Theorem 8.1). Moreover, all the results of the paper apply equally well to the category of differential graded algebras over a commuative ground ring k. Our proofs all adapt essentially verbatim, or else one can use that the homotopy theory of dgas over k is equivalent to that of algebras over the Eilenberg-MacLane ring spectrum Hk (this is proven in [S]).

1.5. **Some background.** Corollary 1.4 was needed in our paper [DS], and we at first believed this result to be obvious. Our attempts to give a careful proof, however, always seemed to fail. A construction of k-invariants for ring spectra had already been given in [L], but that construction seems not well suited for the above classification questions.

Eventually we discovered [BDG], which applied the Dwyer-Kan moduli space technology to the classification of Postnikov extensions in a related context. It will be clear to the reader that the basic methods in the present paper are heavily influenced by [BDG]. However, in order to carry out the [BDG] program we have had to straighten out many points about ring spectra along the way. One of the main things the reader will find here is a new, functorial construction of k-invariants for ring spectra. We also provide a careful proof of a Blakers-Massey theorem for ring spectra.

1.6. **Notation and terminology.** If $\mathcal{C}$ is a category then we write $\mathcal{C}(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. If $\mathcal{C}$ is a model category then $\underline{\mathcal{C}}(X, Y)$ denotes a homotopy function complex from X to Y. The phrase 'homotopy function complex' indicates a construction which has the correct homotopy type even if X is not cofibrant and Y is not fibrant. To fix a particular construction, we use the hammock localization of [DK2].

If R is a commutative ring spectrum then there are model category structures on R-Mod and $R - \mathcal{A}lg$ provided by [SS1]; in each case the fibrations and weak equivalences are determined by the forgetful functor to symmetric spectra. We use these model categories throughout the paper.

To every category $\mathcal{C}$ one can associated a simplicial set $|\mathcal{C}|$ by taking its nerve. In this paper we often abbreviate $|\mathcal{C}|$ to just $\mathcal{C}$, letting the application of the nerve be clear from context.

Finally, if X is a spectrum then $\pi_* X$ always refers to the derived homotopy groups (i.e., homotopy groups of a fibrant replacement).

2. Background on ring spectra

In this section we give some of the basic constructions and properties of ring spectra which will be used throughout the paper.

2.1. **Postnikov sections.** Let R be a connective, commutative ring spectrum. For any R-module V, we let $T_R(V)$ denote the tensor algebra on V. For any pointed simplicial set K, let $T_R(K)$ be shorthand for $T_R(R \wedge \Sigma^\infty K)$. Note that one has maps $T_R(\partial\Delta^n) \to T_R(\Delta^n)$, and these are cofibrations of R-algebras.

If E is a cofibrant, connective R-algebra and

$$\begin{array}{ccc} T_R(\partial\Delta^{n+1}) & \longrightarrow & E \\ \downarrow & & \downarrow \\ T_R(\Delta^{n+1}) & \longrightarrow & E' \end{array}$$

is a pushout diagram of R-algebras, one verifies that $\pi_i(E) \to \pi_i(E')$ is an isomorphism for $i \leq n-1$ (see Lemma A.1). Here we need that E is cofibrant as an R-module (which follows from being cofibrant as an R-algebra, by [SS1, 4.1(3)]) to ensure that the pushout has the correct homotopy type—see the proof of Lemma A.1 for the details.

If E is a cofibrant, connective R-algebra, let $P_n(E)$ be the result of applying the small object argument to E with respect to the set of maps $T_R(\partial\Delta^i) \to T_R(\Delta^i)$

for all $i \geq n+2$ together with the generating trivial cofibrations for $R - \mathcal{A}lg$ from [HSS, 5.4.3] or [SS1, 4.1]. This is similar to the functorial construction of a Postnikov section for differential graded algebras given in [DS, 3.2]. One checks that $P_n(E)$ is fibrant, $\pi_i P_n(E) = 0$ for $i > n$, and $\pi_i E \to \pi_i P_n(E)$ is an isomorphism for $i \leq n$. Also, one has natural maps $P_{n+1}(E) \to P_n(E)$ which are compatible with $E \to P_n(E)$ as n varies.

Finally, if E is a connective R-algebra then we will write $P_n(E)$ as shorthand for $P_n(cE)$, where $cE \to E$ is a fixed functorial cofibrant-replacement for E. Note that one does not have a map $E \to P_n(E)$ in this general case, only a zig-zag $E \xleftarrow{\sim} cE \to P_n(cE) = P_n(E)$.

2.2. **Pushouts of ring spectra.** For the result below we will need to use relative homotopy groups $\pi_*(B, A)$ where $A \to B$ is a map of spectra. Note that we are not assuming that $A \to B$ is a cofibration, as is often done. What we mean by $\pi_*(B, A)$ is therefore $\pi_*(W, A)$, where we have functorially factored $A \to B$ as $A \rightarrowtail W \xrightarrow{\sim} B$.

At many places in the paper we will use the following important result:

Proposition 2.3. *Let R be a connective, commutative ring spectrum and let $n \geq 1$. Suppose given a homotopy pushout square of R-algebras*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in which the following conditions hold:

(i) A is connective;
(ii) $\pi_i(X, A) = 0$ for $i < 2$; and
(iii) $\pi_i(B, A) = 0$ for $i < n+1$.

Then $\pi_i(B, A) \to \pi_i(Y, X)$ is an isomorphism for $i \leq n+1$. In particular, this means $\pi_i(Y, X) = 0$ for $i < n+1$, which implies $\pi_i X \to \pi_i Y$ is an isomorphism for $i < n$ and a surjection for $i = n$.

This result should be a special case of a general Blakers-Massey theorem for ring spectra; however, we don't have a reference for such a result. See [GH, 2.3.13], though, for a full statement in a related context.

For completeness we have included a proof of this result, since we have not been able to find one in the literature. This can be found in Appendix A.

2.4. **Relative homotopy groups.** Let $A \to B$ be a map of R-algebras. We claim that $\pi_*(B, A)$ is in a natural way a bimodule over $\pi_* A$. To explain this, it suffices to assume that A and B are fibrant R-algebras, hence fibrant as spectra. Let $K \hookrightarrow L$ be a cofibration of pointed simplicial sets where K is weakly equivalent to $\partial\Delta^n$ and L is contractible. Then the relative homotopy group $\pi_n(B, A)$ may be described as equivalence classes of diagrams $\mathcal{D}$ of the form

$$\begin{array}{ccc} \Sigma^\infty K & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Sigma^\infty L & \longrightarrow & B, \end{array}$$

where two diagrams $\mathcal{D}$ and $\mathcal{D}'$ are equivalent if there is a diagram

$$\begin{array}{ccc} \Sigma^\infty[(K \times \Delta^1)/(* \times \Delta^1)] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Sigma^\infty[(L \times \Delta^1)/(* \times \Delta^1)] & \longrightarrow & B \end{array}$$

which restricts to $\mathcal{D}$ and $\mathcal{D}'$ under the inclusions $\{0\} \hookrightarrow \Delta^1$ and $\{1\} \hookrightarrow \Delta^1$, respectively.

Suppose given an element $\alpha \in \pi_n(B, A)$, represented by a diagram $\mathcal{D}$ of the form

$$\begin{array}{ccc} \Sigma^\infty(\partial\Delta^n) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Sigma^\infty(\Delta^n) & \longrightarrow & B, \end{array}$$

as above. Also assume given an element $\beta \in \pi_k A$, represented by a map $\Sigma^\infty(\partial\Delta^{k+1}) \to A$. Then one forms the new diagram

$$\begin{array}{ccccccccc} \Sigma^\infty(\partial\Delta^n \wedge \partial\Delta^{k+1}) & \xrightarrow{\cong} & \Sigma^\infty(\partial\Delta^n) \wedge \Sigma^\infty(\partial\Delta^{k+1}) & \longrightarrow & A \wedge A & = & A \wedge A & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty(\Delta^n \wedge \partial\Delta^{k+1}) & \xrightarrow{\cong} & \Sigma^\infty(\Delta^n) \wedge \Sigma^\infty(\partial\Delta^{k+1}) & \longrightarrow & B \wedge A & \xrightarrow{\mu} & B \wedge B & \longrightarrow & B \end{array}$$

which represents a homotopy element of $\pi_{n+k}(B, A)$ (where we are taking $K = \partial\Delta^n \wedge \partial\Delta^{k+1}$ and $L = \Delta^n \wedge \partial\Delta^{k+1}$). Here we would have been slightly better off if we were working with topological spaces rather than simplicial sets, as one can choose homeomorphisms $S^{n-1} \wedge S^k \cong S^{n+k-1}$ and $D^n \wedge S^k \cong D^{n+k}$; but everything works out simplicially as well, with only a little extra care.

We have just described a pairing $\pi_n(B, A) \times \pi_k A \to \pi_{n+k}(B, A)$, and one checks that this makes $\pi_n(B, A)$ into a right module over $\pi_k A$. A similar construction works for the left module structure, and the verification that this gives a bimodule is routine.

Note that in the long exact homotopy sequence of a pair, the connecting homomorphism $\partial\colon \pi_n(B, A) \to \pi_{n-1}(A)$ is a map of $\pi_* A$ bimodules. This is because ∂ sends a homotopy element represented by a diagram

$$\begin{array}{ccc} \Sigma^\infty(\partial\Delta^n) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Sigma^\infty(\Delta^n) & \longrightarrow & B \end{array}$$

to the element $\Sigma^\infty(\partial\Delta^n) \to A$.

3. k-invariants for ring spectra: an outline

In this section we give a basic outline of how k-invariants work for ring spectra. A k-invariant gives rise to a Postnikov extension, and given a Postnikov extension we explain how to construct an associated k-invariant. These basic constructions will then be analyzed in a more sophisticated way later in the paper.

3.1. **Extensions of ring spectra.** We continue to assume that R is a connective, commutative ring spectrum. Fix an $n \geq 1$. Let C be a connective R-algebra such that $P_{n-1}C \simeq C$ and let M be a $\pi_0(C)$-bimodule—that is, a $(\pi_0 C) \otimes_{\pi_0(R)} (\pi_0 C)^{op}$-module. As mentioned in the introduction, we wish to consider ring spectra Y together with a map $Y \to C$ such that

(i) $P_n Y \simeq Y$,
(ii) $P_{n-1}Y \to P_{n-1}C$ is a weak equivalence,
(iii) $\pi_n(Y) \cong M$ as $\pi_0(Y)$-bimodules (where M becomes a $\pi_0(Y)$-bimodule via the isomorphism $\pi_0(Y) \to \pi_0(C)$).

The map $Y \to C$ is called a **Postnikov extension of C** of type (M, n). Note that a particular choice of isomorphism $\pi_n(Y) \cong M$ is not part of the data.

Let $\mathcal{M}(C + (M, n))$ denote the category whose objects are such Postnikov extensions; here a map from $X \to C$ to $Y \to C$ is a weak equivalence $X \to Y$ making the evident triangle commute. We'll call this category the **moduli space** of Postnikov extensions of C of type (M, n).

If $\mathcal{C}$ is a category, we'll write $\pi_0(\mathcal{C})$ for the connected components of the nerve of $\mathcal{C}$. We wish to study $\pi_0\mathcal{M}(C + (M, n))$, as this will tell us how many 'homotopically different' extensions of C there are of type (M, n).

Definition 3.2. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories will be called a **weak equivalence** if it induces a weak equivalence on the nerves. The functor $F\colon \mathcal{C} \to \mathcal{D}$ will be called a **homotopy equivalence** if there is a funtor $G\colon \mathcal{D} \to \mathcal{C}$ and zig-zags of natural transformations between $F \circ G$ and $Id_{\mathcal{D}}$, and between $G \circ F$ and $Id_{\mathcal{C}}$.*

Proposition 3.3. *Suppose $C \to C'$ is a weak equivalence of R-algebras. Then there is an induced functor $\rho\colon \mathcal{M}(C + (M, n)) \to \mathcal{M}(C' + (M, n))$ and this is a homotopy equivalence.*

Proof. Given an object $X \to C'$ of $\mathcal{M}(C' + (M, n))$, functorially factor this map as $X \stackrel{\sim}{\rightarrowtail} X_1 \twoheadrightarrow C'$. Let $\phi(X) = C \times_{C'} X_1$. Then $\phi(X) \to C$ is in $\mathcal{M}(C + (M, n))$, using right properness of $R - \mathcal{A}lg$. So this defines a functor $\phi\colon \mathcal{M}(C' + (M, n)) \to \mathcal{M}(C + (M, n))$.

It is simple to check that there is a zig-zag of natural weak equivalences between the composite $\rho \circ \phi$ and the identity map, and the same for the other composite $\phi \circ \rho$. □

By the above proposition, we can assume that C is a cofibrant R-algebra when studying $\mathcal{M}(C + (M, n))$. We will always be clear about when we are making this assumption, however.

3.4. **Bimodules.** By a C-bimodule we mean a left $(C \wedge_R C^{op})$-module. As remarked in the introduction, there is a C-bimodule HM satisfying $\pi_i(HM) = 0$ for $i \neq 0$ and $\pi_0(HM) \cong M$ (as $\pi_0(C)$-bimodules). Moreover, a typical obstruction theory argument shows that any two such bimodules are weakly equivalent. We abbreviate HM to just M in the rest of the paper.

Remark 3.5. Note that the notion of C-bimodule depends on more than just the homotopy type of C. For if $C \to C'$ is a weak equivalence of R-algebras, the induced map $C \wedge_R C \to C' \wedge_R C'$ need not be a weak equivalence anymore. For this reason we will sometimes have to assume that C is cofibrant as an R-module when dealing with bimodules.

3.6. **Extensions via pullbacks.** If $\mathcal{M}$ is a model category and $X \in \mathcal{M}$, let $\mathcal{M}_{/X}$ denote the usual overcategory whose objects are maps $Y \to X$ in $\mathcal{M}$. Recall that $\mathcal{M}_{/X}$ inherits a model structure from $\mathcal{M}$ in which a map from $Y \to X$ to $Y' \to X$ is a cofibration (respectively fibration, weak equivalence) if and only if the map $Y \to Y'$ is a cofibration (respectively fibration, weak equivalence) in $\mathcal{M}$.

We regard $C \vee \Sigma^{n+1} M$ as an object in $R-\mathcal{A}lg_{/C}$ via the projection $C \vee \Sigma^{n+1} M \to C$. Note that $C \vee \Sigma^{n+1} M$ is actually a *pointed* object of $R-\mathcal{A}lg_{/C}$, since it comes equipped with the evident inclusion $C \hookrightarrow C \vee \Sigma^{n+1} M$.

Suppose given a homotopy class in $\operatorname{Ho}(R-\mathcal{A}lg_{/C})(C, C \vee \Sigma^{n+1} M)$. This can be represented by a map

$$\alpha \colon cC \to f(C \vee \Sigma^{n+1} M)$$

where cC is a cofibrant-replacement of C and $f(C \vee \Sigma^{n+1} M)$ is a fibrant-replacement of $C \vee \Sigma^{n+1} M$ in $R-\mathcal{A}lg_{/C}$. Consider the homotopy fiber of α in $R-\mathcal{A}lg_{/C}$. This is the same as the homotopy pullback of

$$cC \to f(C \vee \Sigma^{n+1} M) \leftarrow C$$

in $R-\mathcal{A}lg$. To be precise, to form this homotopy pullback we functorially factor the maps as

$$cC \overset{\sim}{\rightarrowtail} (cC)' \twoheadrightarrow f(C \vee \Sigma^{n+1} M), \qquad C \overset{\sim}{\rightarrowtail} C' \twoheadrightarrow f(C \vee \Sigma^{n+1} M)$$

and then the homotopy pull back Y is the pullback

$$\begin{array}{ccc} Y & \dashrightarrow & C' \\ \vdots & & \downarrow \\ (cC)' & \longrightarrow & f(C \vee \Sigma^{n+1} M) \end{array}$$

in $R-\mathcal{A}lg$. As pullbacks in $R-\mathcal{A}lg$ are the same as pullbacks in ordinary spectra, it is easy to analyze the homotopy groups of Y. One sees immediately that $\pi_i((cC)', Y) = 0$ for $i \neq n+1$ and $\pi_{n+1}((cC)', Y) \cong M$. So $\pi_i(Y) = 0$ for $i > n$, $\pi_i Y \to \pi_i(cC)'$ is an isomorphism for $i < n$, and the map $\partial \colon \pi_{n+1}((cC)', Y) \to \pi_n Y$ is an isomorphism. By the remarks at the end of Section 2.4, ∂ is an isomorphism of $\pi_0(Y)$-bimodules. Moreover, the map $\pi_{n+1}((cC)', Y) \to \pi_{n+1}(f(C \vee \Sigma^{n+1} M), C)$ is an isomorphism of $\pi_0(Y)$-bimodules, and the codomain of this map is clearly isomorphic to M as a bimodule. We have therefore shown that Y is a Postnikov extension of $(cC)'$ of type (M, n). As we have a map $(cC)' \to f(C \vee \Sigma^{n+1} M) \to C$ which is a weak equivalence, this is also a Postnikov extension of C of type (M, n).

The above remarks give us a function

$$PB \colon \operatorname{Ho}(R-\mathcal{A}lg_{/C})(C, C \vee \Sigma^{n+1} M) \to \pi_0 \mathcal{M}(C + (M, n)).$$

It is clearly not injective, for the following reason. An automorphism $\sigma \colon M \to M$ of $\pi_0(C)$-bimodules induces an automorphism of ring spectra $\sigma \colon C \vee \Sigma^{n+1} M \to C \vee \Sigma^{n+1} M$. If a given homotopy class $\alpha \in \operatorname{Ho}(R-\mathcal{A}lg_{/C})(C, C \vee \Sigma^{n+1} M)$ is composed with this σ, it gives rise to a weakly equivalent pullback.

Let $\operatorname{Aut}(M)$ be the group of $\pi_0(C)$-bimodule automorphisms of M. One way to rephrase the above paragraph is to say that we have an action of $\operatorname{Aut}(M)$ on the set of homotopy classes we're considering, and we get an induced map

$$\widetilde{PB} \colon [\operatorname{Ho}(R-\mathcal{A}lg_{/C})(C, C \vee \Sigma^{n+1} M)] / \operatorname{Aut}(M) \to \pi_0 \mathcal{M}(C + (M, n)).$$

Our main goal in this paper is to show that this map is an isomorphism. Along the way, however, we will actually describe the entire homotopy type of $\mathcal{M}(C+(M,n))$ as opposed to just π_0.

3.7. **k-invariants.** Let $f\colon Y \to C$ be a Postnikov extension of type (M,n). We wish to show that it's in the image of $\widetilde{PB}$. We'll now give a rough outline of how to go about this, which will then be 'cleaned up' in the later sections of the paper.

First of all, we can assume Y is a cofibrant R-algebra (otherwise we replace it with one). Let D be the homotopy pushout $C \amalg^h_Y C$ of $C \xleftarrow{f} Y \xrightarrow{f} C$ in $R-\mathcal{A}lg_{/C}$. This means we factor f as

$$Y \rightarrowtail C' \xrightarrow{\sim} C$$

and we let D be the pushout

$$\begin{array}{ccc} Y & \rightarrowtail & C' \\ \downarrow & & \vdots \\ C' & \cdots\rightarrow & D \end{array}$$

in $R-\mathcal{A}lg$. Note that there is a map $D \to C$ induced by the universal property of pushouts. Applying Proposition 2.3, we have that $\pi_i(C',Y) \to \pi_i(D,C')$ is an isomorphism for $i \leq n+1$. It follows that $\pi_i C' \to \pi_i D$ is an isomorphism for $i \leq n$, and $\pi_{n+1}(D,C') \cong M$ as $\pi_0(C')$-bimodules. (For the 'bimodule' aspect of the last claim, one again uses the remarks in Section 2.4).

Let $E = P_{n+1}D$ (and note that D is cofibrant, so that we have a natural map $D \to P_{n+1}D$). We will later show that the map $C' \to E$ is weakly equivalent in $R-\mathcal{A}lg_{/C}$ to the standard inclusion $C \hookrightarrow C \vee \Sigma^{n+1}M$. This can easily be done by an obstruction theory argument (see also Remark 4.1). After choosing such a weak equivalence, we have that the composite map

$$C' \to D \to E$$

represents an element of $\mathrm{Ho}\,(R-\mathcal{A}lg_{/C})(C, C\vee\Sigma^{n+1}M)$. We will show that choosing a different weak equivalence only affects this element up to the action of $\mathrm{Aut}(M)$, so that we have a well-defined invariant in $[\mathrm{Ho}\,(R-\mathcal{A}lg_{/C})(C, C\vee\Sigma^{n+1}M)]/\,\mathrm{Aut}(M)$. Some unpleasant checking is then required to verify that we have indeed produced an inverse to $\widetilde{PB}$. To organize this 'checking', it helps to rephrase everything in terms of categories—this is what we do in the next section.

4. Categories of k-invariants and Eilenberg-MacLane objects

Let R, C, M, and n be as in the previous section. Our goal for the remainder of the paper is to analyze the homotopy type of the moduli space $\mathcal{M}(C+(M,n))$. To do this we need to introduce some auxilliary categories.

Define the category $\mathcal{E}_C(M,n)$ of ***C*-Eilenberg-MacLane objects** of type (M,n) as follows. The objects of $\mathcal{E}_C(M,n)$ are maps $B \to E$ in $R-\mathcal{A}lg_{/C}$ such that

(i) $B \to C$ is a weak equivalence,
(ii) $B \to E$ becomes a weak equivalence after applying P_n,
(iii) $P_{n+1}E \simeq E$,
(iv) $\pi_{n+1}(E) \cong M$ as $\pi_0(B)$-bimodules (where M becomes a $\pi_0(B)$-bimodule via the isomorphism $\pi_0(B) \to \pi_0(C)$).

A map from $[B \to E]$ to $[B' \to E']$ in this category consists of weak equivalences $B \to B'$ and $E \to E'$ in $R - \mathcal{A}lg_{/C}$ making the evident square commute.

Remark 4.1. Although it is not entirely obvious, we will see later that every object of $\mathcal{E}_C(M,n)$ is weakly equivalent to $C \hookrightarrow C \vee \Sigma^{n+1}M$. This follows from Proposition 4.4(b) below, which shows that $\mathcal{E}_C(M,n)$ is connected.

Likewise, we define the category $\mathcal{K}_C(M,n)$ of **generalized k-invariants for C of type (M,n)**. The objects of $\mathcal{K}_C(M,n)$ are pairs of maps $A \to E \leftarrow B$ in the category $R - \mathcal{A}lg_{/C}$ such that

(i) $A \to C$ is a weak equivalence,
(ii) $B \to E$ is an object in $\mathcal{E}_C(M,n)$.

A map from $A \to E \leftarrow B$ to $A' \to E' \leftarrow B'$ is a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & E & \longleftarrow & B \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ A' & \longrightarrow & E' & \longleftarrow & B'. \end{array}$$

in $R - \mathcal{A}lg_{/C}$ in which all the vertical maps are weak equivalences.

Note that there is a forgetful functor $\mathcal{K}_C(M,n) \to \mathcal{E}_C(M,n)$ which forgets the object A and the map $A \to E$.

Proposition 4.2. *Suppose $C \to C'$ is a weak equivalence of R-algebras. Then there are induced functors $\rho\colon \mathcal{K}_C(M,n) \to \mathcal{K}_{C'}(M,n)$ and $\rho\colon \mathcal{E}_C(M,n) \to \mathcal{E}_{C'}(M,n)$, and both are homotopy equivalences.*

Proof. We will prove the result for $\mathcal{K}_C(M,N)$ and leave the $\mathcal{E}_C(M,n)$ case to the reader. Given $A \to E \leftarrow B$ in $\mathcal{K}_{C'}(M,n)$, produce functorial factorizations $A \overset{\sim}{\rightarrowtail} A' \twoheadrightarrow C'$, $B \overset{\sim}{\rightarrowtail} B' \twoheadrightarrow C'$, and $E \overset{\sim}{\rightarrowtail} E' \twoheadrightarrow C'$. So we have the diagram

$$\begin{array}{ccccc} A & \longrightarrow & E & \longleftarrow & B \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ A' & \longrightarrow & E' & \longleftarrow & B'. \end{array}$$

Define $\phi\colon \mathcal{K}_{C'}(M,n) \to \mathcal{K}_C(M,n)$ by sending $A \to E \leftarrow B$ to the sequence of maps $A' \times_{C'} C \to E' \times_{C'} C \leftarrow B' \times_{C'} C$. It follows by right-properness of $R - \mathcal{A}lg$ that this sequence indeed lies in $\mathcal{K}_C(M,n)$.

Just as in Proposition 3.3, it is simple to produce a zig-zag of natural weak equivalences between $\rho \circ \phi$ and the identity, and the same for $\phi \circ \rho$. $\square$

Our next goal is to show that the category $\mathcal{K}_C(M,n)$ is weakly equivalent to the moduli space $\mathcal{M}(C + (M,n))$. First, observe that there is a functor $PB_C\colon \mathcal{K}_C(M,n) \to \mathcal{M}(C + (M,n))$ which sends $A \to E \leftarrow B$ to its homotopy pullback in $R - \mathcal{A}lg_{/C}$. As in Section 3.6, whenever we talk about the 'homotopy pullback' $A \times^h_E B$ of a diagram $A \to E \leftarrow B$ we mean the pullback of $A' \to E \leftarrow B'$ where we have functorially factored $A \to E$ and $B \to E$ as trivial cofibrations followed by fibrations

$$A \overset{\sim}{\rightarrowtail} A' \twoheadrightarrow E \qquad \text{and} \qquad B \overset{\sim}{\rightarrowtail} B' \twoheadrightarrow E.$$

Note that there is a natural map from the pullback of $A \to E \leftarrow B$ to its homotopy pullback.

To verify that the image of PB_C actually lands in $\mathcal{M}(C + (M, n))$, first recall that pullbacks of R-algebras are the same as pullbacks of spectra. This immediately verifies conditions (i) and (ii) in the definition of $\mathcal{M}(C+(M,n))$ (Section 3.1). For the third condition, let $P = A' \times_E B'$. Note that in the long exact homotopy sequence of a pair the connecting homomorphism $\pi_{n+1}(A', P) \to \pi_n(P)$ is an isomorphism since $P_{n-1}A' \simeq A'$, and in fact it is an isomorphism of $\pi_0(P)$-bimodules by the discussion in Section 2.4. But we also have an isomorphism of $\pi_0(P)$-bimodules $\pi_{n+1}(A', P) \to \pi_{n+1}(E, B')$, as well as an isomorphism of $\pi_0(B')$-bimodules $\pi_{n+1}(E) \to \pi_{n+1}(E, B')$. For the fact that these are bimodule maps, we again refer to Section 2.4. Since $\pi_{n+1}(E, B') \cong \pi_{n+1}(E, B) \cong M$, we find that $\pi_n(P)$ is isomorphic to M as $\pi_0(P)$-bimodules.

Proposition 4.3. *The functor $PB_C : \mathcal{K}_C(M, n) \to \mathcal{M}(C+(M, n))$ is a weak equivalence.*

The proof is somewhat long, and will be given in Section 5. The basic idea is to try to construct a homotopy inverse functor $k : \mathcal{M}(C + (M, n)) \to \mathcal{K}_C(M, n)$. This will be our **generalized k-invariant**. Given an $X \to C$ in $\mathcal{M}(C + (M, n))$ such that X is cofibrant, construct a homotopy pushout

$$\begin{array}{ccc} X & \longrightarrow & P_{n-1}X \\ \downarrow & & \downarrow g \\ P_{n-1}X & \xrightarrow{h} & Z \end{array}$$

and consider the maps

$$P_{n-1}X \xrightarrow{h} P_{n+1}Z \xleftarrow{g} P_{n-1}X.$$

One can check that this gives an element $k(X)$ in $\mathcal{K}_{P_{n+1}(P_{n-1}C)}(M, n)$. Since $P_{n+1}(P_{n-1}C) \simeq C$ this is 'almost' an element of $\mathcal{K}_C(M, n)$. Some care is required in getting around this small difference, and this is part of what is accomplished in Section 5.

The above proposition reduces the problem of studying $\mathcal{M}(C + (M, n))$ to that of studying $\mathcal{K}_C(M, n)$. We do this by analyzing the forgetful functor $\mathcal{K}_C(M, n) \to \mathcal{E}_C(M, n)$. We will prove the following in Section 7:

Proposition 4.4.

(a) There is a homotopy fiber sequence of spaces

$$\underline{R - \mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1}M) \to |\mathcal{K}_C(M, n)| \to |\mathcal{E}_C(M, n)|,$$

where the first term denotes the homotopy function complex in the model category $R - \mathcal{A}lg_{/C}$.

(b) There is a weak equivalence of spaces $|\mathcal{E}_C(M, n)| \simeq B\,\mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the group of automorphisms of M as a $\pi_0(C)$-bimodule.

Part (a) is basically a routine 'moduli space' problem, of the type considered in [BDG, Section 2]. Part (b) involves similar techniques but also requires some careful manipulations of ring spectra.

5. The moduli space of k-invariants

In this section we will prove Proposition 4.3. As mentioned above, our first hope would be to construct a generalized k-invariant functor $k\colon \mathcal{M}(C+(M,n)) \to \mathcal{K}_C(M,n)$ to provide a homotopy inverse for $PB_C\colon \mathcal{K}_C(M,n) \to \mathcal{M}(C+(M,n))$. This doesn't quite work out. Instead, we restrict to the case where C is cofibrant and construct a functor $k\colon \mathcal{M}(C+(M,n)) \to \mathcal{K}_{C'}(M,n)$ where $C' = P_{n+1}(P_{n-1}C)$. We then use this to show that PB_C is a weak equivalence.

For the rest of this section we assume that C is a cofibrant R-algebra. Throughout the following, let $C' = P_{n+1}(P_{n-1}C)$. Next we define $\mathcal{D}(C,M,n)$, a category of diagrams which will be useful in defining a generalized k-invariant functor $k\colon \mathcal{M}(C+(M,n)) \to \mathcal{K}_{C'}(M,n)$.

The objects of $\mathcal{D}(C,M,n)$ are the commutative diagrams D of the form

$$\begin{array}{ccc} H & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

in which $A \to C$ and $B \to C$ are weak equivalences after applying P_{n-1}, and $H \to C$ lies in $\mathcal{M}(C+(M,n))$. The morphisms in $\mathcal{D}(C,M,n)$ are the maps of commuting diagrams. Using the weak equivalence $C \to P_{n-1}C \to P_{n+1}(P_{n-1}C) = C'$, we will produce a diagram

$$\begin{array}{ccccc} \mathcal{A}(D) & \longrightarrow & \mathcal{E}(D) & \longleftarrow & \mathcal{B}(D) \\ & \searrow & \downarrow & \swarrow & \\ & & C' & & \end{array}$$

which is functorial in $\mathcal{D}(C,M,n)$ and has the following properties:

(1) The diagram $\mathcal{A}(D) \to \mathcal{E}(D) \leftarrow \mathcal{B}(D)$ lies in $\mathcal{K}_{C'}(M,n)$, where we have used the weak equivalence $C \to C'$ to make M a bimodule over $\pi_0(C')$.

(2) There is a natural zig-zag of weak equivalences in $R-\mathcal{A}lg_{/C'}$ between H and the homotopy pullback of $\mathcal{A}(D) \to \mathcal{E}(D) \leftarrow \mathcal{B}(D)$.

(3) If $A \twoheadrightarrow E \twoheadleftarrow B$ is an object in $\mathcal{K}_C(M,n)$ in which the indicated maps are fibrations, and D is the diagram

$$\begin{array}{ccc} A \times_E B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C, \end{array}$$

then there is a natural zig-zag of weak equivalences between $A \to E \leftarrow B$ and $\mathcal{A}(D) \to \mathcal{E}(D) \leftarrow \mathcal{B}(D)$ in $R-\mathcal{A}lg_{/C'}$.

(4) Suppose D' is another diagram

$$\begin{array}{ccc} H' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C \end{array}$$

in $\mathcal{D}(C,M,n)$, and assume there is a map of diagrams $D \to D'$ which is the identity on C, a weak equivalence on $H \to H'$ and on $B \to B'$, and a weak equivalnce after applying P_{n-1} to $A \to A'$. Then the induced maps $\mathcal{A}(D) \to \mathcal{A}(D')$, $\mathcal{E}(D) \to \mathcal{E}(D')$, and $\mathcal{B}(D) \to \mathcal{B}(D')$ are weak equivalences.

In a moment we will explain how to construct $\mathcal{A}(D)$, $\mathcal{E}(D)$, and $\mathcal{B}(D)$, and we will verify the above properties. But first we show how this implies what we want.

Proof of Proposition 4.3. Using Propositions 3.3 and 4.2, it suffices to analyze the case when C is a cofibrant R-algebra.

Suppose given $X \to C$ in $\mathcal{M}(C + (M, n))$. We define $k(X) \in \mathcal{K}_{C'}(M, n)$ to be $\mathcal{A}(D) \to \mathcal{E}(D) \leftarrow \mathcal{B}(D)$ where D is the diagram

$$\begin{array}{ccc} X & \longrightarrow & C \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle Id} \\ X & \longrightarrow & C. \end{array}$$

We think of $k(X)$ as the **generalized $\boldsymbol{k}$-invariant of $\boldsymbol{X}$**. It gives a functor $k\colon \mathcal{M}(C + (M, n)) \to \mathcal{K}_{C'}(M, n)$.

We now have the following (non-commutative) diagram of functors:

$$\begin{array}{ccc} \mathcal{M}(C + (M, n)) & \xrightarrow{\rho_1} & \mathcal{M}(C' + (M, n)) \\ {\scriptstyle PB_C}\uparrow & \searrow^{k} & \uparrow{\scriptstyle PB_{C'}} \\ \mathcal{K}_C(M, n) & \xrightarrow[\rho_2]{} & \mathcal{K}_{C'}(M, n) \end{array}$$

The maps labelled ρ_1 and ρ_2 are known to be weak equivalences, by Propositions 3.3 and 4.2. If we can show that there is a zig-zag of natural transformations between ρ_1 and the composite $PB_{C'} \circ k$, as well as between ρ_2 and the composite $k \circ PB_C$, it will follow that all maps in the diagram induce isomorphisms on the homotopy groups of the nerves—so all the maps will be weak equivalences.

Now, property (2) says precisely that there is a zig-zag of natural weak equivalences between $PB_{C'} \circ k$ and ρ_1. So we consider the other composite. Let $A \to E \leftarrow B$ be an object in $\mathcal{K}_C(M, n)$. Functorially factor $A \to E$ and $B \to E$ as

$$A \overset{\sim}{\rightarrowtail} A' \twoheadrightarrow E \qquad \text{and} \qquad B \overset{\sim}{\rightarrowtail} B' \twoheadrightarrow E$$

and let $H = A' \times_E B'$. So $PB_C(A \to E \leftarrow B) = H$. Let D_1, D_2, and D_3 be the following three squares:

$$\begin{array}{ccc} H & \longrightarrow & C \\ \downarrow & & \downarrow \\ H & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} H & \longrightarrow & C \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} H & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C. \end{array}$$

Note that there are natural transformations $D_1 \to D_2$ and $D_3 \to D_2$. By property (4), we get the following chain of equivalences:

$$\begin{array}{ccccc} \mathcal{A}(D_1) & \longrightarrow & \mathcal{E}(D_1) & \longleftarrow & \mathcal{B}(D_1) \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ \mathcal{A}(D_2) & \longrightarrow & \mathcal{E}(D_2) & \longleftarrow & \mathcal{B}(D_2) \\ \sim\uparrow & & \sim\uparrow & & \sim\uparrow \\ \mathcal{A}(D_3) & \longrightarrow & \mathcal{E}(D_3) & \longleftarrow & \mathcal{B}(D_3) \end{array}$$

Note that the top row is $k(H) = k(PB_C(A \to E \leftarrow B))$. Using property (3), there is a natural zig-zag of weak equivalences between the last row and the diagram $A' \to E \leftarrow B'$, which in turn is weakly equivalent to $A \to E \leftarrow B$. So we have established our zig-zag of natural transformations between $k \circ PB_C$ and ρ_2. This completes the proof. □

Finally we are reduced to doing some actual work: we must construct the functors $\mathcal{A}$, $\mathcal{E}$, and $\mathcal{B}$. Suppose given a diagram D in $\mathcal{D}(C, M, n)$ of the form

$$\begin{array}{ccc} H & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C. \end{array}$$

Recall that $A \to C$ and $B \to C$ are weak equivalences after applying P_{n-1}, and $H \to C$ lies in $\mathcal{M}(C + (M, n))$. Let $cX \xrightarrow{\sim} X$ be a functorial cofibrant-replacement in $R - \mathcal{A}lg$. Consider the composites $cH \to P_{n-1}(cH) \to P_{n-1}(cA)$ and $cH \to P_{n-1}(cH) \to P_{n-1}(cB)$: functorially factor them as

$$cH \rightarrowtail SA \overset{\sim}{\twoheadrightarrow} P_{n-1}(cA) \qquad \text{and} \qquad cH \rightarrowtail SB \overset{\sim}{\twoheadrightarrow} P_{n-1}(cB).$$

We obtain a diagram

$$\begin{array}{ccccc} SA & \longrightarrow & P_{n+1}[(SA) \amalg_{cH} (SB)] & \longleftarrow & SB \\ & \searrow^{\sim} & \downarrow & \swarrow^{\sim} & \\ & & P_{n+1}(P_{n-1}C). & & \end{array}$$

We let $\mathcal{A}(D) = SA$, $\mathcal{E}(D) = P_{n+1}[(SA) \amalg_{cH} (SB)]$, $\mathcal{B}(D) = SB$, and $C' = P_{n+1}(P_{n-1}C)$.

Notice the following:

- Property (4) follows immediately from our definitions.
- There is a natural map from cH into the pullback $\mathcal{A}(D) \times_{\mathcal{E}(D)} \mathcal{B}(D)$, and of course a natural map from the pullback to the homotopy pullback. This gives a natural zig-zag
$$H \xleftarrow{\sim} cH \to \mathcal{A}(D) \times^h_{\mathcal{E}(D)} \mathcal{B}(D)$$
in $R - \mathcal{A}lg_{/C'}$.

In order to check the remaining properties we will need the following two lemmas.

Lemma 5.1. *Fix $n \geq 1$, let W be a connective R-algebra such that $P_{n-1}W \simeq W$, and let M be a $\pi_0(W)$-bimodule. Let $A \leftarrow X \to B$ be maps in $R - \mathcal{A}lg_{/W}$ where $A \to W$ and $B \to W$ are weak equivalences and $X \to W$ is in $\mathcal{M}(W + (M, n))$. Let P denote the homotopy pushout $A \amalg^h_X B$. Then $\pi_{n+1}P$ is isomorphic to $\pi_n X$ as a $\pi_0(X)$-bimodule.*

Proof. Consider the map $f \colon \pi_{n+1}(A, X) \to \pi_{n+1}(P, B)$, which is an isomorphism by Proposition 2.3. This is readily seen to be a map of $\pi_0(X)$-bimodules, using the observations from Section 2.4. Here we regard $\pi_{n+1}(P, B)$ as a $\pi_0(X)$-bimodule via the map of $\pi_0(R)$-algebras $\pi_0(X) \to \pi_0(B)$. The map $\pi_{n+1}(P) \to \pi_{n+1}(P, B)$ is a map of $\pi_0(B)$-bimodules (and hence $\pi_0(X)$-bimodules, by restriction) which is an isomorphism by our assumptions on B.

Finally, the connecting homomorphism $\pi_{n+1}(A, X) \to \pi_n(X)$ is a map of $\pi_0(X)$-bimodules which is an isomorphism by our assumptions on X and A. So we have established that $\pi_{n+1}(P) \cong \pi_n(X)$ as $\pi_0(X)$-bimodules. □

Lemma 5.2. *Let $n \geq 1$ and let W, M be as in the previous lemma.*

(a) Suppose that $A \twoheadrightarrow E \twoheadleftarrow B$ is in $\mathcal{K}_W(M, n)$ (where the indicated maps are fibrations), and let $H = A \times_E B$. Then the induced map $A \amalg^h_H B \to E$ becomes a weak equivalence after applying P_{n+1}.

(b) Suppose given cofibrations $A \leftarrowtail X \rightarrowtail B$ in $R - \mathcal{A}lg_{/W}$ such that both $A \to W$ and $B \to W$ are weak equivalences, and where $P_n X \simeq X$. Also assume both $P_{n-1}X \to P_{n-1}A$ and $P_{n-1}X \to P_{n-1}B$ are weak equivalences. Then the diagram $A \to P_{n+1}(A \amalg_X B) \leftarrow B$ lies in $\mathcal{K}_{P_{n+1}W}(\pi_n X, n)$.

(c) Again suppose given cofibrations $A \leftarrowtail X \rightarrowtail B$ in $R{-}\mathcal{A}lg_{/W}$ satisfying the same hypotheses as in (b). Let X' be the homotopy pullback of $A \to P_{n+1}(A \amalg_X B) \leftarrow B$. Then the induced map $X \to X'$ is a weak equivalence.

Proof. The statements in (a) and (c) follow directly from Proposition 2.3. One should note that pullbacks (and homotopy pullbacks) in the category of R-algebras are the same as those in the category of symmetric spectra.

For the statement in (b), one uses Proposition 2.3 together with Lemma 5.1 above. □

Proof of Properties (1)–(3). Properties (1) and (2) follow directly from Lemma 5.2 parts (b) and (c), respectively. So we turn to property (3).

Let $A \twoheadrightarrow E \twoheadleftarrow B$ be an object in $\mathcal{K}_C(M, n)$, and let H be the pullback $A \times_E B$. Let $\mathcal{D}$ be the diagram

$$\begin{array}{ccc} H & \longrightarrow & B\,. \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & C \end{array}$$

Functorially factor the maps $cH \to cA$ and $cH \to cB$ as $cH \rightarrowtail S'A \xrightarrow{\sim} cA$ and $cH \rightarrowtail S'B \xrightarrow{\sim} cB$. Note that one gets induced maps $S'A \to SA$ and $S'B \to SB$ (where SA and SB appeared in our construction of $\mathcal{A}(D)$, etc.), and these maps are weak equivalences. Let $\mathcal{E}'(D) = P_{n+1}(S'A \amalg_{cH} S'B)$, so that there is an induced map $\mathcal{E}'(D) \to \mathcal{E}(D)$.

Notice that we have a map $S'A \amalg_{cH} S'B \to cA \amalg_{cH} cB \to cE$, and therefore get an induced map $\mathcal{E}'(D) \to P_{n+1}(cE)$. This is a weak equivalence by Lemma 5.2(a).

Now we have the following:

$$\begin{array}{ccccc}
SA & \longrightarrow & \mathcal{E}(D) & \longleftarrow & SB \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
S'A & \longrightarrow & \mathcal{E}'(D) & \longleftarrow & S'B \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
cA & \longrightarrow & P_{n+1}(cE) & \longleftarrow & cB \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
cA & \longrightarrow & cE & \longleftarrow & cB \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
A & \longrightarrow & E & \longleftarrow & B.
\end{array}$$

So we have obtained a natural zig-zag of weak equivalences between the diagrams $\mathcal{A}(D) \to \mathcal{E}(D) \leftarrow \mathcal{B}(D)$ and $A \to E \leftarrow B$. □

6. Non-unital algebras

Before proceeding further with our main results we need to develop a little machinery. This concerns non-unital C-algebras and their relations to C-bimodules. We first discuss a model structure on non-unital C-algebras. Then we define an 'indecomposables' functor from non-unital C-algebras to C-bimodules and study its interaction with Postnikov stages.

Let R be a commutative S-algebra and C an R-algebra. Define a **non-unital *C*-algebra** to be a non-unital monoid in the category of C-bimodules, that is, an algebra over the monad

$$\widetilde{T}_C(M) = M \amalg (M \wedge_C M) \amalg (M \wedge_C M \wedge_C M) \amalg \cdots$$

in C-bimod. Let $\mathcal{N}onU_C$ denote the category of non-unital C-algebras.

A map of non-unital C-algebras is defined to be a *fibration* or a *weak equivalence* if the underlying map in C-bimod (or R-Mod) is a fibration or a weak equivalence. A map is then a *cofibration* if it has the left lifting property with respect to all trivial fibrations. Below we will use [SS1] to verify that this gives a model structure on $\mathcal{N}onU_C$.

First recall that C-bimod is just another name for the category $C \wedge_R C^{op}$-Mod. The model structure on R-modules lifts to a model structure on C-bimodules by [SS1, 4.1]. Let $F_C\colon R\text{-Mod} \to \text{C-bimod}$ be the left adjoint to the forgetful functor C-bimod $\to R$-Mod. The generating trivial cofibrations in C-bimod are of the form $F_C(j)\colon C \wedge_R K \wedge_R C^{op} \to C \wedge_R L \wedge_R C^{op}$ where $j\colon K \to L$ is a generating trivial cofibration in R-Mod.

Theorem 6.1. *The above notions of cofibration, fibration, and weak equivalence form a cofibrantly generated model category structure on $\mathcal{N}onU_C$. The generating cofibrations and trivial cofibrations are of the form $\widetilde{T}_C(F_C(K)) \to \widetilde{T}_C(F_C(L))$ where $K \to L$ is a generating cofibration or trivial cofibration in R-Mod.*

Proof. To establish the model structure on $\mathcal{N}onU_C$ we modify the arguments for unital monoids in [SS1, 6.2]. The argument in [SS1] is mostly formal except for one key step. For us, this step is to show that given a generating trivial cofibration $K \to L$ in C-bimod, the pushout in $\mathcal{N}onU_C$ of the diagram

$$\begin{array}{ccc} \widetilde{T}_C(K) & \longrightarrow & \widetilde{T}_C(L) \\ \downarrow & & \\ X & & \end{array}$$

is the colimit $P = \operatorname{colim} P_n$ in C-bimod of a sequence

$$X = P_0 \to P_1 \to \cdots \to P_n \to \cdots$$

where P_n is obtained from P_{n-1} by a pushout in C-bimod. We then show that the monoid axiom implies that $X = P_0 \to P$ is a weak equivalence. From this it follows directly from [SS1, 2.3(1)] that the given model structure exists on $\mathcal{N}onU_C$.

To construct P_n from P_{n-1}, we replace the functor W in [SS1, 6.2] by a functor

$$W\colon \mathcal{P}(\underline{n}) \to C\text{-bimod}$$

where $\mathcal{P}(\underline{n})$ is the poset category of subsets of $\underline{n} = \{1, 2, \cdots, n\}$ and inclusions. For $S \subseteq \underline{n}$, define

$$W(S) = (C \vee X) \wedge_C B_1 \wedge_C (C \vee X) \wedge_C B_2 \wedge_C \cdots \wedge_C B_n \wedge_C (C \vee X)$$

with

$$B_i = \begin{cases} K & \text{if } i \notin S \\ L & \text{if } i \in S \end{cases}.$$

Let Q_n be the colimit of $W(S)$ over $\mathcal{P}(\underline{n}) - \underline{n}$ (that is, the proper subsets of $\underline{n}$). As in [SS1, 6.2], one has maps $Q_n \to P_{n-1}$ and $Q_n \to W(\underline{n})$ and defines P_n as the following pushout in C-bimod

$$\begin{array}{ccc} Q_n & \longrightarrow & W(\underline{n}) \\ \downarrow & & \downarrow \\ P_{n-1} & \longrightarrow & P_n \end{array}$$

Set $P = \operatorname{colim} P_n$, the colimit in C-bimod. Arguments analogous to those in [SS1, 6.2] show that P is naturally a non-unital C-algebra and has the universal property of the pushout of $X \leftarrow \widetilde{T}_C(K) \to \widetilde{T}_C(L)$ in non-unital C-algebras.

We next show that the monoid axiom for R-Mod implies that each map $P_{n-1} \to P_n$ is a weak equivalence whenever $K \to L$ is a generating trivial cofibration in C-bimod. These generating trivial cofibrations are of the form $F_C(K') \to F_C(L')$ where $K' \to L'$ is a generating trivial cofibration in R-Mod.

Since pushouts in C-bimod are created in R-Mod and R-Mod is symmetric monoidal, we consider the pushouts defining P_n in R-Mod. Replacing $K \to L$ by $F_C(K') \to F_C(L')$, we see that $Q_n \to W(\underline{n})$ is isomorphic to

$$Q'_n \wedge_R (C \vee X)^{\wedge_R (n)} \to (L')^{\wedge_R n} \wedge_R (C \vee X)^{\wedge_R (n)}$$

where $Q'_n \to (L')^{\wedge_R n}$ is the n-fold box product of $K' \to L'$. The pushout product axiom implies that $Q'_n \to (L')^{\wedge_R n}$ is a trivial cofibration. The monoid axiom then implies that the pushouts $P_{n-1} \to P_n$ are weak equivalences and $X = P_0 \to \operatorname{colim} P_n$ is a weak equivalence. $\square$

There is a functor $K\colon \mathcal{N}on U_C \to (C \downarrow R-\mathcal{A}lg \downarrow C)$ which takes a non-unital algebra N to $K(N) = (C \xrightarrow{\eta} C \vee N \xrightarrow{\pi} C)$ where η and π are the obvious inclusion and projection. This has a right adjoint

$$I\colon (C \downarrow R-\mathcal{A}lg \downarrow C) \to \mathcal{N}on U_C$$

called the *augmentation ideal* functor. The functor I sends $(C \to X \to C)$ to the fiber of $X \to C$.

Proposition 6.2. *The functors (K, I) form a Quillen equivalence*

$$K\colon \mathcal{N}on U_C \xrightarrow{\sim} (C \downarrow R-\mathcal{A}lg \downarrow C)$$

Proof. The same statement for commutative ring spectra is proved in [B, 2.2]. The proof works verbatim in the non-commutative case; see also [BM, Thm. 8.6] for a vast generalization. □

6.3. **Indecomposables.** Our next task is to compare non-unital C-algebras with C-bimodules. The *indecomposables* functor $Q\colon \mathcal{N}on U_C \to$ C-bimod takes X in $\mathcal{N}on U_C$ to the pushout of $* \leftarrow X \wedge X \to X$. Its right adjoint Z: C-bimod $\to \mathcal{N}on U_C$ sends a bimodule to itself equipped with the zero product.

Proposition 6.4. *The functors Q and Z form a Quillen pair:*

$$\mathcal{N}on U_C \underset{Z}{\overset{Q}{\rightleftarrows}} C\text{-bimod}$$

Proof. The functor Z obviously preserves fibrations and trivial fibrations. Again, see also [B, 3.1] and [BM, Prop. 8.7] for similar statements. □

If N is a C-bimodule, it is easy to see that $Q(ZN) \cong N$. When we consider the derived functors $\bar{Q}(\bar{Z}N)$ the situation changes, however. Here we must take a cofibrant replacement of ZN before applying Q. It turns out that $\bar{Q}(\bar{Z}N)$ typically has nonzero homotopy groups in infinitely many dimensions, even if N did not.

If $n \geq 1$ and N has no homotopy groups in dimensions smaller than n, then the same turns out to be true for $\bar{Q}(\bar{Z}N)$. Morover, the nth homotopy group of $\bar{Q}(\bar{Z}N)$ is easy to analyze, and it is the same as that of N. This is the content of Proposition 6.5 below.

Before stating the proposition we need a couple of pieces of new notation. We'll use c and f to denote cofibrant- and fibrant-replacement functors in a model category, and we leave it to the reader to decide from context which model category the replacements are taking place in. In the statement of the proposition below, for instance, the c is being applied in $\mathcal{N}on U_C$ and the f's are being applied in C-bimod.

Also, we note that the P_n's in the statement of the proposition refer to Postnikov sections in the category of C-bimodules. These are constructed analogously to those for ring spectra, but here one forms pushouts with respect to the maps $(C \wedge_R C^{op}) \wedge \partial\Delta^i \to (C \wedge_R C^{op}) \wedge \Delta^i$ for $i > n+1$.

Proposition 6.5. *Fix $n \geq 1$, and let N be a C-bimodule such that $\pi_i(N) = 0$ for $i < n$. There is a natural weak equivalence of C-bimodules*

$$P_n[QcZ(fN)] \to P_n(fN).$$

Proof. First note that there are natural maps $QcZ(fN) \to QZ(fN) \to fN$; applying P_n to this composite gives the map in the statement of the proposition. Call this map g.

Recall that $\widetilde{T}_C : C\text{-bimod} \to \mathcal{N}on U_C$ is the left adjoint of the forgetful functor. Note that this is a left Quillen functor, and that there are natural isomorphisms $Q\widetilde{T}_C(W) \cong W$ by an easy adjointness argument.

If K is a spectrum, write FK as shorthand for $(C \wedge_R C^{op}) \wedge K$. This is the free C-bimodule generated by K.

Write $\mathbb{P}_n$ for the Postnikov section functor in the category $\mathcal{N}on U_C$. The construction is the same as for ring spectra, but in this case we form pushouts with respect to the maps $\widetilde{T}_C(F\partial\Delta^i) \to \widetilde{T}_C(F\Delta^i)$ for $i > n+1$. Using that Q is a left adjoint and therefore preserves pushouts and colimits and that $Q\widetilde{T}_C(W) \cong W$, one can show that $Q(\mathbb{P}_n X)$ is obtained from QX by forming pushouts with respect to $F(\partial\Delta^i) \to F(\Delta^i)$ for $i > n+1$. Given the description above of Postnikov sections for C-bimodules, this implies that the map $QX \to Q(\mathbb{P}_n X)$ induces a weak equivalence

$$P_n[QX] \to P_n[Q\mathbb{P}_n X]. \tag{6.6}$$

Without loss of generality we may assume that N is a cofibrant C-bimodule. Consider the natural map of non-unital algebras $\widetilde{T}_C(N) \to Z(N)$, adjoint to the isomorphism $N \to UZ(N)$ where $U : \mathcal{N}on U_C \to C\text{-bimod}$ is the forgetful functor. Using our hypothesis on N, one finds that $N \wedge_C N \wedge_C \cdots \wedge_C N$ doesn't have any homotopy groups in dimensions less than $2n$ as long as there are at least two smash factors of N. It follows immediately that the induced map $\mathbb{P}_n[\widetilde{T}_C(N)] \to \mathbb{P}_n(ZN)$ is a weak equivalence.

Consider the trivial fibration $cZ(fN) \to Z(fN)$. Since $\widetilde{T}_C(N)$ is cofibrant (since N is), the map $\widetilde{T}_C(N) \to Z(N) \to Z(fN)$ lifts to a map $\widetilde{T}_C(N) \to cZ(fN)$. This becomes a weak equivalence after applying $\mathbb{P}_n$, by the previous paragraph.

In the square

$$\begin{array}{ccc} P_nQ[\widetilde{T}_C N] & \longrightarrow & P_nQ[\mathbb{P}_n(\widetilde{T}_C N)] \\ \downarrow & & \downarrow \\ P_nQ[cZ(fN)] & \longrightarrow & P_nQ[\mathbb{P}_n(cZ(fN))] \end{array}$$

the two horizontal maps are weak equivalences by (6.6). The previous paragraph shows that the right vertical map is a weak equivalence, so the left vertical map is as well. Thus, we have an equivalence

$$P_n(N) \cong P_n[Q\widetilde{T}_C(N)] \to P_nQ[cZ(fN)].$$

It is easy to use the adjoint functors to see that the composite of this map with our map $g : P_nQ[cZ(fN)] \to P_n(fN)$ is the map $P_nN \to P_n(fN)$ induced by $N \to fN$. Since this composite is a weak equivalence, so is the map g. □

Proposition 6.7. *Fix $n \geq 1$, and let N be a C-bimodule such that $\pi_i(N) = 0$ for $i \neq n$. If $X \in \mathcal{N}on U_C$ is weakly equivalent to $Z(N)$, then the natural map $cX \to ZP_n[QcX]$ is a weak equivalence.*

Proof. The natural map in the statement is the composite $\eta : cX \to ZQ[cX] \to ZP_n[QcX]$. Since there will necessarily be a weak equivalence $cX \to Z(fN)$, it suffices to check that η is a weak equivalence when $X = Z(fN)$. In this case we consider the diagram

$$\begin{array}{ccccc} cZ(fN) & \longrightarrow & ZQ[cZ(fN)] & \longrightarrow & ZP_nQ[cZ(fN)] \\ & \overset{\sim}{\searrow} & \downarrow & & \downarrow \\ & & Z(fN) & \longrightarrow & ZP_n(fN). \end{array}$$

The composite of the top horizontal maps is η, the vertical maps come from the counit of the (Q, Z) adjunction and the diagram is readily checked to commute. The bottom horizontal map is a weak equivalence because $fN \to P_n(fN)$ is a weak equivalence, and Z preserves all weak equivalences. Finally, we know by the preceding proposition that the right vertical map is a weak equivalence, so η is a weak equivalence as well. □

7. The moduli space of Eilenberg-MacLane objects

Recall where we are at this point in the paper. We have completed the proof of Proposition 4.3, and our next goal is to prove Proposition 4.4.

7.1. **General moduli space technology.** If $\mathcal{C}$ is a model category and X is an object of $\mathcal{C}$, let $\mathcal{M}_{\mathcal{C}}(X)$ denote the category consisting of all objects weakly equivalent to X, where the maps are weak equivalences. This is called the **Dwyer-Kan classification space** of X, or the **moduli space** of X. It is a theorem of Dwyer-Kan [DK3, 2.3] that $|\mathcal{M}_{\mathcal{C}}(X)| \simeq B\,\mathrm{hAut}(X)$ where $\mathrm{hAut}(X)$ denotes the simplicial monoid of homotopy automorphisms of X. This is simply the subcomplex of the homotopy function complex $\underline{\mathcal{M}}(X, X)$ consisting of all path components which are invertible in the monoid $\pi_0\underline{\mathcal{M}}(X, X)$.

If X and Y are two objects of $\mathcal{C}$ then we'll write $\mathcal{H}om_{\mathcal{C}}(X, Y)$ for the category consisting of diagrams

$$X \xleftarrow{\sim} U \longrightarrow V \xleftarrow{\sim} Y$$

where the indicated maps are weak equivalences. A morphism in this category is a natural weak equivalence between diagrams which is the identity on X and Y. It is another result of Dwyer-Kan that one has a natural zig-zag of weak equivalences between $\mathcal{H}om_{\mathcal{C}}(X, Y)$ and the homotopy function complex $\underline{\mathcal{C}}(X, Y)$. This follows from [DK1, 6.2(i), 8.4].

When Y is fibrant one may consider a simpler moduli space. Namely, let $\mathcal{H}om_{\mathcal{C}}(X, Y)^f$ be the category whose objects are diagrams

$$X \xleftarrow{\sim} U \longrightarrow Y.$$

A map in this category is again a weak equivalence of diagrams which is the identity on X and Y. There is an inclusion functor $\mathcal{H}om_{\mathcal{C}}(X, Y)^f \to \mathcal{H}om_{\mathcal{C}}(X, Y)$, and it is stated in [BDG, 2.7] that this is a weak equivalence when Y is fibrant. For a proof, see [D2].

Remark 7.2. All of the Dwyer-Kan theorems we mentioned above were actually proven only for *simplicial* model categories. Using the main result of [D1], however, they can be immediately extended to all combinatorial model categories. All of the model categories considered in the present paper are combinatorial, so we will freely make use of this technology.

7.3. **Applications to ring spectra.** Note that there is a functor

$$\Theta\colon \mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, C\vee\Sigma^{n+1}M) \to \mathcal{K}_C(M,n)$$

which sends a diagram

$$C \xleftarrow{\sim} U \longrightarrow V \xleftarrow{\sim} C\vee\Sigma^{n+1}M$$

in $R-\mathcal{A}lg_{/C}$ to the object of $\mathcal{K}_C(M,n)$ represented by

$$U \to V \leftarrow C$$

(where the second map is the composite $C \hookrightarrow C\vee\Sigma^{n+1}M \xrightarrow{\sim} V$).

The following result is very similar to [BDG, 2.11].

Lemma 7.4. *The sequence of maps*

$$\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, C\vee\Sigma^{n+1}M) \to \mathcal{K}_C(M,n) \to \mathcal{M}_{R-\mathcal{A}lg_{/C}}(C)\times\mathcal{E}_C(M,n)$$

is a homotopy fiber sequence of simplicial sets.

Note that $\mathcal{M}_{R-\mathcal{A}lg_{/C}}(C)$ is contractible, as C is a terminal object of this category. So this lemma, together with the identification of $\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(X,Y)$ with the homotopy function complex $\underline{R-\mathcal{A}lg_{/C}}(X,Y)$, yields Proposition 4.4(a).

Proof. The second map is the functor $F\colon \mathcal{K}_C(M,n)\to\mathcal{M}_{R-\mathcal{A}lg_{/C}}(C)\times\mathcal{E}_C(M,n)$ which sends an object $A\to E\leftarrow B$ to the pair consisting of A and $E\leftarrow B$. The proof of this lemma will be an application of Quillen's Theorem B—however, a little care is required.

Let $\mathcal{K}^f_C(M,n)$ denote the full subcategory of $\mathcal{K}_C(M,n)$ consisting of objects $A\to E\leftarrow B$ where the maps $E\to C$ and $B\to C$ are fibrations. Let $\mathcal{E}^f_C(M,n)$ denote the analogous subcategory of $\mathcal{E}_C(M,n)$. The inclusions $\mathcal{K}^f_C(M,n)\hookrightarrow\mathcal{K}_C(M,n)$ and $\mathcal{E}^f_C(M,n)\hookrightarrow\mathcal{E}_C(M,n)$ are readily checked to be homotopy equivalences. Let $\widetilde{F}\colon \mathcal{K}^f_C(M,n)\to\mathcal{E}^f_C(M,n)$ be the restriction of the functor F.

To apply Quillen's Theorem B [Q], we must check that for every map $[E'\leftarrow B']\to[E''\leftarrow B'']$ in $\mathcal{E}^f_C(M,n)$ the induced map of overcategories

$$(\widetilde{F}\downarrow[E'\leftarrow B'])\to(\widetilde{F}\downarrow[E''\to B''])$$

is a weak equivalence. There is a functor

$$\phi'\colon \mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C,E')^f\to(\widetilde{F}\downarrow[E'\leftarrow B'])$$

sending a diagram $C\xleftarrow{\sim}A\to E'$ to the pair consisting of the object $A\to E'\leftarrow B'$ in $\mathcal{K}^f_C(M,n)$ together with the identity map from $\widetilde{F}(A\to E'\leftarrow B')$ to $[E'\leftarrow B']$. This functor ϕ' is readily checked to be a homotopy equivalence. Similarly, one has

$$\phi''\colon \mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C,E'')^f\to(\widetilde{F}\downarrow[E''\leftarrow B''])$$

which is a homotopy equivalence by the same argument. The map

$$\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C,E')^f\to\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C,E'')^f$$

is a weak equivalence because it is naturally equivalent to the map of function complexes $\underline{R-\mathcal{A}lg_{/C}}(C,E')\to\underline{R-\mathcal{A}lg_{/C}}(C,E'')$ (which is itself a weak equivalence because $E'\to E''$ is a weak equivalence). So we have established that our map of overcategories is a weak equivalence.

Quillen's Theorem B now tells us that the sequence

$$(\widetilde{F} \downarrow [f(C \vee \Sigma^{n+1} M) \leftarrow C]) \to \mathcal{K}_C^f(M,n) \xrightarrow{\widetilde{F}} \mathcal{E}_C^f(M,n)$$

is a homotopy fiber sequence, where the basepoint in the base space is taken to be the object $[f(C \vee \Sigma^{n+1} M) \leftarrow C]$. We have already remarked that the overcategory appearing here is homotopy equivalent to $\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, f(C \vee \Sigma^{n+1} M))^f$, and that the inclusions $\mathcal{K}_C^f(M,n) \hookrightarrow \mathcal{K}_C(M,n)$ and $\mathcal{E}_C^f(M,n) \hookrightarrow \mathcal{E}_C(M,n)$ are homotopy equivalences. So to complete the proof of the lemma it suffices to note two things. First, we have the commutative square

$$\begin{array}{ccc}
\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, f(C \vee \Sigma^{n+1} M)) & \longrightarrow & \mathcal{K}_C(M,n) \\
\uparrow \sim & & \uparrow \sim \\
\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, f(C \vee \Sigma^{n+1} M)) & \longrightarrow & \mathcal{K}_C^f(M,n)
\end{array}$$

where both vertical maps are the inclusions and the horizontal maps are induced by the functor Θ defined in Section 7.3. Second, one has a commutative triangle

$$\begin{array}{ccc}
\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, f(C \vee \Sigma^{n+1} M)) & \longrightarrow & \mathcal{K}_C(M,n) \\
\sim \downarrow & \nearrow & \\
\mathcal{H}om_{R-\mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1} M) & &
\end{array}$$

where, again, all the maps are the obvious ones. □

Our next goal is to identify $|\mathcal{E}_C(M,n)|$ as in Proposition 4.4(b). We first show that for C a cofibrant R-algebra, $\mathcal{E}_C(M,n)$ is equivalent to a moduli space in the category $\mathcal{N}onU_C$ of non-unital C-algebras. This then reduces further to a moduli space in the category of C-bimodules. Computations in the category of C-bimodules are relatively simple, so that it is not hard to identify the homotopy type of the moduli space in C-bimodules as $B\,\mathrm{Aut}(M)$. Finally, we remove the cofibrancy condition in Corollary 7.8.

Lemma 7.5. *Assume C is a cofibrant R-algebra. There are weak equivalences of categories*

$$\mathcal{E}_C(M,n) \simeq \mathcal{M}_{(C \downarrow R-\mathcal{A}lg \downarrow C)}(C \vee \Sigma^{n+1} M) \simeq \mathcal{M}_{\mathcal{N}onU_C}(\Sigma^{n+1} M).$$

Proof. We write $\mathcal{E}$ for $\mathcal{E}_C(M,n)$. The argument proceeds in several steps. First, let $\mathcal{E}'$ be the full subcategory of $\mathcal{E}$ whose objects $B \to E$ have $B = C$ (and the map $B \to C$ the identity). Let $\mathcal{E}''$ be the full subcategory of $\mathcal{E}$ whose objects are maps $B \rightarrowtail E$ in $R-\mathcal{A}lg_{/C}$ in which both B and E are cofibrant R-algebras and $B \to E$ is a cofibration. Finally, let $\mathcal{E}'''$ be the full subcategory of $\mathcal{E}'$ whose objects are in both $\mathcal{E}'$ and $\mathcal{E}''$. Notice that there is a chain of inclusions

$$\mathcal{E} \hookleftarrow \mathcal{E}'' \hookleftarrow \mathcal{E}''' \hookrightarrow \mathcal{E}'.$$

We claim that each of these inclusions induces a weak equivalence on nerves. This is easy for $\mathcal{E}'' \hookrightarrow \mathcal{E}$ and $\mathcal{E}''' \hookrightarrow \mathcal{E}'$, just using functorial factorizations.

Define a functor $\theta \colon \mathcal{E}'' \to \mathcal{E}'''$ by sending an object $B \rightarrowtail E$ in $\mathcal{E}''$ to the object $C \rightarrowtail C \amalg_B E$. To see that this lies in $\mathcal{E}'''$ one can use Ken Brown's lemma [Ho1] to show that pushing out a weak equivalence along a cofibration yields another weak equivalence, provided all the domains and codomains of the original maps are cofibrant. It is simple to see that θ gives a homotopy inverse for the inclusion $\mathcal{E}''' \hookrightarrow \mathcal{E}''$.

If $C \to X$ lies in $\mathcal{E}'$, a straightforward argument shows that X is weakly equivalent to $C \vee \Sigma^{n+1}M$ in the category $(C \downarrow R{-}\mathcal{A}lg \downarrow C)$ (basically, one uses obstruction theory to directly construct a zig-zag of weak equivalences). So $\mathcal{E}'$ is simply the moduli space $\mathcal{M}_{(C\downarrow R-\mathcal{A}lg\downarrow C)}(C \vee \Sigma^{n+1}M)$.

Recall from Proposition 6.2 that there is a Quillen equivalence

$$\mathcal{N}onU_C \xrightarrow{\sim} (C \downarrow R - \mathcal{A}lg \downarrow C)$$

in which the right adjoint I sends $C \to X \to C$ to the fiber of $X \to C$. Note that $I(C \vee \Sigma^{n+1}M) \simeq \Sigma^{n+1}M$ where $\Sigma^{n+1}M$ is given the trivial structure of non-unital C-algebra (in which the product is zero). Thus, this Quillen equivalence implies that $\mathcal{M}_{(C\downarrow R-\mathcal{A}lg\downarrow C)}(C \vee \Sigma^{n+1}M) \simeq \mathcal{M}_{\mathcal{N}onU_C}(\Sigma^{n+1}M)$. □

We next reduce from $\mathcal{N}onU_C$ to C-bimod:

Proposition 7.6. *The functor Z induces a weak equivalence $\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \to \mathcal{M}_{\mathcal{N}onU_C}(\Sigma^{n+1}M)$.*

Proof. Since Z preserves all weak equivalences, it induces a functor between moduli categories in the obvious way. Consider the composite functor $\mathcal{N}onU_C \to C$-bimod given by

$$X \mapsto P_{n+1}(Q(cX)),$$

where c is any cofibrant-replacement functor in $\mathcal{N}onU_C$. Applying Proposition 6.5 with n replaced by $n+1$, we know that if X is weakly equivalent to $Z(\Sigma^{n+1}M)$ then $P_{n+1}(QcX)$ is a C-bimodule whose only non-vanishing homotopy group lies in dimension $n+1$ and is isomorphic to M. So $P_{n+1}Qc$ induces a functor $F\colon \mathcal{M}_{NonU_C}(\Sigma^{n+1}M) \to \mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M)$.

Proposition 6.5 implies that there is a natural zig-zag of weak equivalences between the composite $F \circ Z$ and the identity functor. Proposition 6.7 implies the same for the composite $Z \circ F$. It follows that the maps induced by Z and F on the nerves of the categories are homotopy inverses. □

We now need to analyze $\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M)$. This is something which boils down to an explicit computation. For the next proposition, recall that to any element $X \in \mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M)$ we may associate its homotopy group $\pi_{n+1}X$ regarded as a $\pi_0(C)$-bimodule. This bimodule is isomorphic to M, and in this way we obtain a functor $\pi_{n+1}\colon \mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$. The codomain is the category of $\pi_0(C)$-bimodules which are isomorphic to M, with maps the isomorphisms; said differently, it is the moduli space in the model category of $\pi_0(C)$-bimodules where the weak equivalences are isomorphisms and where every map is both a cofibration and a fibration. We use this functor in Corollary 7.8 and Theorem 8.1 to identify the action of $\mathrm{Aut}(M)$.

Proposition 7.7. *Assume that C is cofibrant as an R-module (e.g., C is a cofibrant R-algebra). Then the functor $\pi_{n+1}\colon \mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$ is a weak equivalence. Consequently, one has $\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \simeq B\,\mathrm{Aut}(M)$ where the automorphism group is taken in the category of $\pi_0(C)$-bimodules.*

Proof. The Dwyer-Kan result [DK3, 2.3] identifies $\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M)$ with the space $B\,\mathrm{hAut}(\Sigma^{n+1}M)$, where hAut denotes the simplicial monoid of homotopy automorphisms in the model category C-bimod. This is the subcomplex of the homotopy function complex $\underline{C\text{-bimod}}(\Sigma^{n+1}M, \Sigma^{n+1}M)$ consisting of all path components which are invertible in π_0. We'll now compute this homotopy function complex.

The model category of C-bimodules is enriched, tensored, and cotensored over symmetric spectra. So for any bimodules N_1 and N_2 there is a symmetric spectrum mapping space $F(N_1, N_2)$. The homotopy function complex is simply the zeroth space $\mathrm{Ev}_0\, F(cN_1, fN_2)$. Also, since C-bimod is a stable model category one has $F(\Sigma cN_1, f\Sigma cN_2) \simeq F(cN_1, fN_2)$. We need to use $F(c\Sigma^{n+1}M, f\Sigma^{n+1}M) \simeq F(cfM, cfM)$.

Now, it is simple to compute that $\pi_i F(cfM, cfM) = 0$ for $i > 0$ and $\pi_0 F(cfM, cfM) = \mathrm{Hom}_{\pi_0(C)}(M, M)$, the group of endomorphisms of M as a $\pi_0(C)$-bimodule. One way to do this is to recall that for any two C-bimodules N_1 and N_2 there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\pi_*(C\wedge_R C^{op})}(\pi_*(N_1), \Sigma^{-q}\pi_*(N_2)) \Rightarrow \pi_{q-p}F(cN_1, fN_2).$$

In the case $N_1 = N_2 = M$ the E_2-term completely vanishes in the range $q \geq 0$ except for the single group $\mathrm{Ext}^0_{\pi_*(C\wedge_R C^{op})}(M, M)$ when $p = q = 0$ (this uses that $C \wedge_R C$ is connective, which in turn uses our cofibrancy assumption on C). This group is the same as $\mathrm{Hom}_{\pi_0(C\wedge_R C^{op})}(M, M)$. Finally, we note that $\pi_0(C \wedge_R C^{op}) \cong \pi_0(C) \otimes_{\pi_0 R} \pi_0(C)^{op}$. This follows from the spectral sequence

$$\mathrm{Tor}^{\pi_* R}_{p,q}(\pi_* C, \pi_* C^{op}) \Rightarrow \pi_{p+q}(C \wedge_R C^{op})$$

(again using our cofibrancy assumption on C), together with the fact that R and C are connective.

Putting this all together, it readily follows that $\mathrm{hAut}(\Sigma^{n+1}M) \simeq \mathrm{Aut}(M)$ (the latter regarded as a discrete group).

Finally, we return to our functor $\pi_{n+1} \colon \mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$. Since the homotopy groups of both the domain and codomain vanish except for π_1, it suffices to show the functor induces an isomorphism on π_1. Note that there are obvious maps

$$\mathrm{Aut}(M) \to \pi_1\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M) \quad \text{and} \quad \mathrm{Aut}(M) \to \pi_1\mathcal{M}_{\pi_0(C)\text{-bimod}}(M).$$

The first, for instance, sends an automorphism σ to the loop represented by the induced map of bimodules $\sigma\colon \Sigma^{n+1}M \to \Sigma^{n+1}M$; the second is defined similarly. These maps obviously commute with the functor π_{n+1}. But the map $\mathrm{Aut}(M) \to \pi_1\mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$ is readily seen to be an isomorphism, and it follows from our analysis of $\mathrm{hAut}(\Sigma^{n+1}M)$ above that the corresponding map $\mathrm{Aut}(M) \to \pi_1\mathcal{M}_{C\text{-bimod}}(\Sigma^{n+1}M)$ is also an isomorphism. This finishes the proof. $\square$

To any object $E \leftarrow B$ in $\mathcal{E}_C(M, n)$ we may associate the abelian group $\pi_{n+1}E$ which will be a $\pi_0(C)$-bimodule via the isomorphism $\pi_0(B) \cong \pi_0(C)$ and the map $\pi_0(B) \to \pi_0(E)$. So we have a functor $\pi_{n+1} \colon \mathcal{E}_C(M, n) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$.

Corollary 7.8. *The functor $\pi_{n+1} \colon \mathcal{E}_C(M, n) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$ is a weak equivalence. Consequently, $\mathcal{E}_C(M, n) \simeq B\,\mathrm{Aut}(M)$.*

Proof. Let $cC \xrightarrow{\sim} C$ be a cofibrant-replacement in the category of R-algebras. By Proposition 4.2 the evident map $\mathcal{E}_{cC}(M,n) \to \mathcal{E}_C(M,n)$ is a weak equivalence. We then have a zig-zag of weak equivalences

$$\mathcal{E}_C(M,n) \simeq \mathcal{E}_{cC}(M,n) \simeq \mathcal{M}_{NonU_{cC}}(\Sigma^{n+1}M) \simeq \mathcal{M}_{cC\text{-bimod}}(\Sigma^{n+1}M)$$

provided by Lemma 7.5 and Proposition 7.6. There is an obvious π_{n+1} functor from each of these categories landing in $\mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$, and the relevant triangles all commute. Since $\pi_{n+1}\colon \mathcal{M}_{cC\text{-bimod}}(\Sigma^{n+1}M) \to \mathcal{M}_{\pi_0(C)\text{-bimod}}(M)$ is a weak equivalence by Proposition 7.7, we deduce that the same is true for the π_{n+1} functor with domain $\mathcal{E}_C(M,n)$. □

We have finally completed our main proof:

Proof of Proposition 4.4. Part (a) follows directly from Lemma 7.4 and the remarks following its proof. Part (b) is a consequence of the preceding corollary. □

8. The main result

Finally, we can pull everything together and prove the main theorem:

Theorem 8.1. *Fix $n \geq 1$. Let R be a connective ring spectrum, and let C be a connective R-algebra such that $P_nC \simeq C$. Let M be a $\pi_0(C)$-bimodule. There is a homotopy fiber sequence of spaces*

$$\underline{R - \mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1}M) \to |\mathcal{M}(C + (M,n))| \to B\operatorname{Aut}(M).$$

Consequently, one has a bijection

$$[Ho\,(R - \mathcal{A}lg_{/C})(C, C \vee \Sigma^{n+1}M)]/\operatorname{Aut}(M) \cong \pi_0\mathcal{M}(C + (M,n))$$

where $\operatorname{Aut}(M)$ *acts on the second factor of* $C \vee \Sigma^{n+1}M$.

Proof. First one uses that $\mathcal{M}(C+(M,n)) \simeq \mathcal{K}_C(M,n)$, from Proposition 4.3. Then one uses the homotopy fiber sequence

$$\mathcal{M}_{R-\mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1}M) \to \mathcal{K}_C(M,n) \to \mathcal{E}_C(M,n)$$

established in Lemma 7.4 and our identification $\mathcal{E}_C(M,n) \simeq B\operatorname{Aut}(M)$. This proves the first claim of the theorem.

To prove the second statement, we look at the long exact homotopy sequence for the above fiber sequence. Since $\mathcal{E}_C(M,n)$ is connected, this identifies $\pi_0\mathcal{K}_C(M,n)$ with a quotient of $\pi_0[\mathcal{M}_{R-\mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1}M)]$ by an action of $\pi_1\mathcal{E}_C(M,n) \cong \operatorname{Aut}(M)$. We must identify the action.

For brevity, write $S = \pi_0[\mathcal{M}_{R-\mathcal{A}lg_{/C}}(C, C \vee \Sigma^{n+1}M)]$. Every equivalence class $s \in S$ can be represented by a diagram

$$C \xleftarrow{\sim} A \xrightarrow{g} f(C \vee \Sigma^{n+1}M) \longleftarrow C \vee \Sigma^{n+1}M.$$

Let $\sigma \in \operatorname{Aut}(M)$. Under the identification $\pi_1\mathcal{E}_C(M,n) \cong \operatorname{Aut}(M)$, σ corresponds to the self-map of the object $[C \hookrightarrow C \vee \Sigma^{n+1}M]$ which is the identity on C and induced by σ on M. We can just as well represent σ as a self-map of $[C \hookrightarrow f(C \vee \Sigma^{n+1}M)]$.

To determine the action of σ on s we do the usual thing: we lift the loop represented by σ to a path in $\mathcal{K}_C(M,n)$ beginning at s, and we take the terminal point of that path. Our path is the map

$$\begin{array}{ccccc} A & \xrightarrow{g} & f(C\vee\Sigma^{n+1}M) & \leftarrowtail & C \\ \| & & \downarrow{\scriptstyle\sigma} & & \| \\ A & \xrightarrow{\sigma g} & f(C\vee\Sigma^{n+1}M) & \leftarrowtail & C \end{array}$$

This identifies the action of $\mathrm{Aut}(M)$ on $S = \mathrm{Ho}\,(R-\mathcal{A}lg_{/C})(C, C\vee\Sigma^{n+1}M)$ with the action coming from the second factor of $C\vee\Sigma^{n+1}M$. □

Remark 8.2. The isomorphism $[\mathrm{Ho}\,(R-\mathcal{A}lg_{/C})(C, C\vee\Sigma^{n+1}M)]/\,\mathrm{Aut}(M) \to \pi_0\mathcal{M}(C+(M,n))$ produced in the theorem is precisely the pullback map $\widetilde{PB}$ defined in Section 3.6. This follows at once by looking at the various maps we used in our identifications (particularly the one of Proposition 4.3).

Remark 8.3. Theorem 1.3 and Corollary 1.4 from the introduction are the special case of the above theorem where $R = S$.

Appendix A. Proof of a Blakers-Massey result

In this section we will give the proof of Proposition 2.3. We start with a lemma. Although the lemma is stated for an arbitrary $(k-1)$-connected cofibration $K \rightarrowtail L$, the main application is when $K \to L$ is $\partial\Delta^k \to \Delta^k$ or a coproduct of such maps. Let $\pi_i^s(L,K)$ denote $\pi_i(\Sigma^\infty L, \Sigma^\infty K)$.

Lemma A.1. *Let R be a connective, commutative ring spectrum and let $k \geq 2$. Let $K \rightarrowtail L$ be a cofibration of pointed simplicial sets such that $\pi_i^s(L,K) = 0$ for $i < k$. Suppose X is a cofibrant, connective R-algebra and*

$$\begin{array}{ccc} T_R(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ T_R(L) & \longrightarrow & Y \end{array}$$

is a pushout square of R-algebras. Then

(a) $\pi_i X \to \pi_i Y$ is an isomorphism for $i < k-1$, and surjective for $i = k-1$.

(b) $\pi_k(Y,X)$ is isomorphic to $[\pi_0(X)\otimes_{\pi_0(R)}\pi_0(X)^{op}]\otimes_{\mathbb{Z}}\pi_k^s(L,K)$. That is to say, it is the free $\pi_0(X)$-bimodule generated by the abelian group $\pi_k^s(L,K)$.

Proof. In [SS1, Proof of Lemma 6.2] the pushout Y is described as a certain directed colimit of pushouts in the category of spectra. If we let $P_0 = X$, then there is a sequence of cofibrations of spectra

$$P_0 \rightarrowtail P_1 \rightarrowtail P_2 \rightarrowtail \cdots$$

whose colimit is the underlying spectrum of Y and where P_r is obtained from P_{r-1} by a pushout diagram of spectra

$$\begin{array}{ccc} W_r \wedge (X\wedge_R X \wedge_R \cdots \wedge_R X) & \longrightarrow & P_{r-1} \\ \downarrow & & \downarrow \\ W_r' \wedge (X\wedge_R X \wedge_R \cdots \wedge_R X) & \longrightarrow & P_r. \end{array}$$

Here there are $r+1$ copies of X in the smash product, and $W_r \to W'_r$ is the r-fold box product of $K \to L$. So $W'_r/W_r \cong (L/K)^{\wedge r}$, which is $(rk-1)$-connected.

Note that $X \wedge_R X \wedge_R \cdots \wedge_R X$ is connective, since both X and R are connective and X is cofibrant. This follows from the fact that $X \wedge_R X$ is the realization of the simplicial spectrum $[n] \mapsto X \wedge R^{\wedge n} \wedge X$, for instance. So we find that the spectrum

$$P_r/P_{r-1} \cong (L/K)^{\wedge r} \wedge (X \wedge_R \wedge \cdots \wedge_R X)$$

has no homotopy groups below dimension rk. Hence, $\pi_j P_{r-1} \to \pi_j P_r$ is an isomorphism for $j < rk-1$ and a surjection for $j = rk-1$. It follows immediately that $\pi_j X \to \pi_j Y$ is an isomorphism for $j < k-1$, and a surjection for $j = k-1$.

The homotopy cofiber sequence $P_1/P_0 \to Y/P_0 \to Y/P_1$ shows that $\pi_k(P_1, P_0) \to \pi_k(Y, X)$ is an isomorphism. This uses $k \geq 2$. But $P_0 \to P_1$ sits in a pushout diagram

$$\begin{array}{ccc} K \wedge (X \wedge_R X) & \longrightarrow & P_0 \\ \downarrow & & \downarrow \\ L \wedge (X \wedge_R X) & \longrightarrow & P_1, \end{array}$$

and so $P_1/P_0 \cong L/K \wedge (X \wedge_R X)$. As $X \wedge_R X$ is connective, one has $\pi_k(P_1, P_0) \cong \pi_k^s(L, K) \otimes_{\mathbb{Z}} \pi_0(X \wedge_R X)$. But using the spectral sequence $\mathrm{Tor}_{p,q}^{\pi_* R}(\pi_* X, \pi_* X) \Rightarrow \pi_{p+q}(X \wedge_R X)$ (or otherwise), it follows that $\pi_0(X \wedge_R X) \cong \pi_0(X) \otimes_{\pi_0(R)} \pi_0(X)$ for X cofibrant.

A little thought shows that the isomorphism

$$\pi_k(Y, X) \cong \pi_k^s(L, K) \otimes_{\mathbb{Z}} [(\pi_0 X) \otimes_{\pi_0 R} (\pi_0 X)^{op}]$$

we've produced is one of $\pi_0(X)$-bimodules. □

Remark A.2. Suppose given a diagram

$$\begin{array}{ccc} \partial\Delta^k & \xrightarrow{f} & K \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & L \end{array}$$

and let α denote the element of $\pi_k(L, K)$ it represents. For any $x, y \in \pi_0(X)$ we then get an element $x \cdot \alpha \cdot y \in \pi_k(Y, X)$, using part (b) of the above lemma. One readily checks that the connecting homomorphism $\pi_k(Y, X) \to \pi_{k-1} X$ takes the element $x \cdot \alpha \cdot y$ to $x \cdot [f] \cdot y$, where the latter refers to the $\pi_0(X)$-bimodule structure on $\pi_{k-1} X$.

Proof of Proposition 2.3. We can assume that A, B, and X are cofibrant and fibrant as R-algebras (hence fibrant as spectra), and that $A \to B$ is a cofibration of R-algebras. Then we can also assume that Y is the pushout of $T_R(L) \leftarrow T_R(K) \to X$ (rather than just the homotopy pushout).

First, we inductively construct a sequence of cofibrations between cofibrant and fibrant objects

$$A = A_0 = A'_0 \rightarrowtail A_1 \rightarrowtail A'_1 \rightarrowtail A_2 \rightarrowtail A'_2 \rightarrowtail \cdots$$

together with compatible maps $A_i \to B$, $A'_i \to B$, with the following properties:

(i) $\pi_i(A'_r) \to \pi_i(A_{r+1})$ is an isomorphism for $i < n+r$ and a surjection for $i = n+r$;

(ii) $\pi_i(A_r) \to \pi_i(A'_r)$ is an isomorphism for $i < n + r$;
(iii) $\pi_i(A_r) \to \pi_i(B)$ is an isomorphism for $i < n + r$;
(iv) $\pi_i(A'_r) \to \pi_i(B)$ is an isomorphism for $i < n+r$ and a surjection for $i = n+r$.

From this we conclude that the map $\operatorname{colim}_r A_r \to B$ is a weak equivalence.

Assuming we have constructed $A'_{k-1} \to B$ as above, we construct the cofibration $A'_{k-1} \rightarrowtail A_k$. Here we want to kill off the kernel of $\pi_{n+k-1}A'_{k-1} \to \pi_{n+k-1}B$. Choose a set of diagrams

$$\begin{array}{ccc} \partial\Delta^{n+k} & \longrightarrow & A'_{k-1} \\ \downarrow & & \downarrow \\ \Delta^{n+k} & \longrightarrow & B \end{array}$$

which generate $\pi_{n+k}(B, A'_{k-1})$, which one can do because A'_{k-1} and B are fibrant. Define Z_k to be the pushout of R-algebras

$$\begin{array}{ccc} \coprod T(\partial\Delta^{n+k}) & \longrightarrow & A'_{k-1} \\ \downarrow & & \downarrow \\ \coprod T(\Delta^{n+k}) & \longrightarrow & Z_k, \end{array}$$

and factor the induced map $Z_k \to B$ as $Z_k \overset{\sim}{\rightarrowtail} A_k \twoheadrightarrow B$. Note that A_k is fibrant and cofibrant. Also, note that Z_k is obtained from A'_{k-1} by pushing out along a map $T(\amalg\partial\Delta^{n+k}) \to T(\amalg\Delta^{n+k})$, so by Lemma A.1 we know $\pi_i A'_{k-1} \to \pi_i A_k$ is an isomorphism for $i < n + k - 1$ and a surjection for $i = n + k - 1$. A simple argument shows that we have killed the image of $\pi_{n+k}(B, A'_{k-1})$ so that $\pi_{n+k-1}A_k \to \pi_{n+k-1}B$ is an isomorphism.

Assuming we have constructed $A_k \to B$ as above, we construct the cofibration $A_k \rightarrowtail A'_k$. Here we want to add homotopy elements so that $\pi_{n+k}A'_k \to \pi_{n+k}B$ is surjective. Choose a set of maps $\Delta^{n+k}/\partial\Delta^{n+k} \to B$ which generate $\pi_{n+k}B$, which we can regard as a collection of diagrams

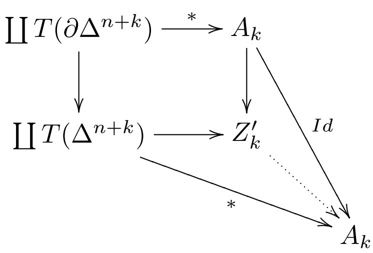
$$\begin{array}{ccc} \partial\Delta^{n+k} & \xrightarrow{\ *\ } & A_k \\ \downarrow & & \downarrow \\ \Delta^{n+k} & \longrightarrow & B. \end{array}$$

Let Z'_k be the pushout

$$\begin{array}{ccc} \coprod T(\partial\Delta^{n+k}) & \xrightarrow{\ *\ } & A_k \\ \downarrow & & \downarrow \\ \coprod T(\Delta^{n+k}) & \longrightarrow & Z'_k, \end{array}$$

and factor the induced map $Z'_k \to B$ as $Z_k \overset{\sim}{\rightarrowtail} A'_k \twoheadrightarrow B$. Note that A'_k is cofibrant and fibrant. By Lemma A.1 we have $\pi_i A_k \to \pi_i A'_k$ is an isomorphism for $i < n+k-1$ and surjective for $i = n + k - 1$. In fact, by examining $\pi_{n+k}(A'_k, A_k) \to \pi_{n+k-1}A_k$ (using Remark A.2) we find it's the zero map—so $\pi_{n+k-1}A_k \to \pi_{n+k-1}A'_k$ is an isomorphism. Moreover, one sees readily that $\pi_{n+k}A'_k \to \pi_{n+k}B$ is surjective.

We make one further remark before proceeding to the next stage of the argument, which is to point out that A_k is a retract of Z'_k. This is because of the diagram



It follows that $\pi_i(A_k) \to \pi_i(A'_k)$ is injective, for all i. We will need this later.

Now that we have constructed A_k and A'_k, we define $X_k = X \amalg_A A_k$ and $X'_k = X \amalg_A A'_k$; so we have two chains

$$\begin{array}{ccccccccc} A & \longrightarrow & A_1 & \longrightarrow & A'_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & X_1 & \longrightarrow & X'_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \end{array}$$

Let $X_\infty = \operatorname{colim}_r X_r$ and $A_\infty = \operatorname{colim}_r A_r$. Note that $X_\infty \cong A_\infty \amalg_A X \simeq B \amalg_A X = Y$. So we must show that $\pi_i(X) \to \pi_i(X_\infty)$ is an isomorphism for $i < n$, a surjection for $i = n$, and that $\pi_{n+1}(A_\infty, A) \to \pi_{n+1}(X_\infty, X)$ is an isomorphism.

Now, A_{k+1} is obtained from A_k via a sequence of cofibrations

$$A_k \rightarrowtail Z'_k \overset{\sim}{\rightarrowtail} A'_k \rightarrowtail Z_{k+1} \overset{\sim}{\rightarrowtail} A_{k+1}$$

where $A_k \to Z'_k$ is a cobase-change of maps $T(\partial\Delta^{n+k}) \to T(\Delta^{n+k})$ where the map $T(\partial\Delta^{n+k}) \to A_k$ is null, and $A'_k \to Z_{k+1}$ is a cobase-change of maps $T(\partial\Delta^{n+k+1}) \to T(\Delta^{n+k+1})$. Since $X = X_0 = X'_0$, $\pi_i(X) \to \pi_i(X_1)$ is an ismorphism for $i < n$ and a surjection for $i = n$ by Lemma A.1. It also follows that for $k \geq 1$, $\pi_i(X_k) \to \pi_i(X_{k+1})$ is an isomorphism for $i < n+k$. We therefore have that $\pi_i(X) \to \pi_i(X_\infty)$ is an isomorphism for $i < n$ and a surjection for $i = n$.

We only have left to show that $\pi_{n+1}(A_\infty, A) \to \pi_{n+1}(X_\infty, X)$ is an isomorphism. Consider the two sequences

$$\begin{array}{ccccccccc} * & \longrightarrow & A_1/A & \longrightarrow & A'_1/A & \longrightarrow & A_2/A & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & X_1/X & \longrightarrow & X'_1/X & \longrightarrow & X_2/X & \longrightarrow & \cdots \end{array}$$

Looking only at the first row, the long exact homotopy sequences for the successive levels give an exact couple. One gets a spectral sequence

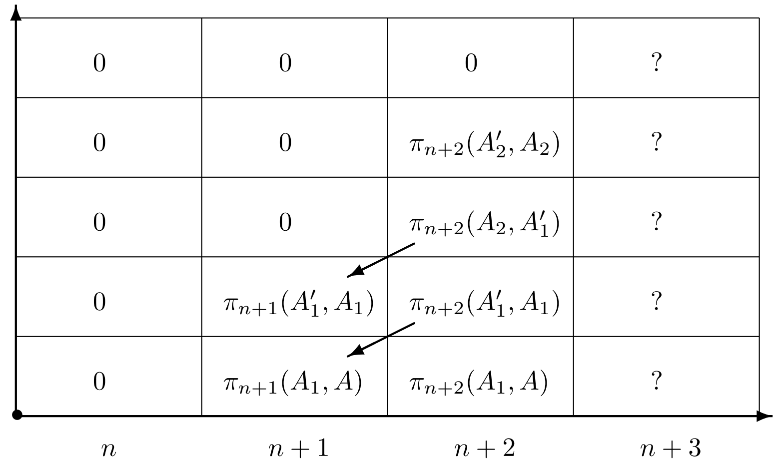
$$E^1_{p,q} = \begin{cases} \pi_p(A_{i+1}, A'_i) & \text{if } q = 2i \\ \pi_p(A'_i, A_i) & \text{if } q = 2i+1 \end{cases}$$

converging to $\pi_p(A_\infty, A)$. Here the indexing is set up so that the spectral sequence is drawn on a grid with $E^1_{p,q}$ in the (p,q) spot, and the differentials have $d_r \colon E^1_{p,q} \to E^1_{p-1,q-r}$. The groups in the $E^\infty_{p,*}$ column are an associated graded of $\pi_p(A_\infty, A)$.

There is a second spectral sequence with all the A's replaced with X's, and the first spectral sequence maps to the second.

Now, from our construction of the A's and X's one checks readily that $\pi_{n+k+1}(A_{k+1}, A'_k) \to \pi_{n+k+1}(X_{k+1}, X'_k)$ is an isomorphism for each k, since we can identify both groups by Lemma A.1. For the same reason, $\pi_{n+k}(A'_k, A_k) \to \pi_{n+k}(X'_k, X_k)$ is an isomorphism.

In the spectral sequence for the A's one finds that the E^1-page looks as follows (with the arrows depicting possible d_1's):



The picture is similar for the spectral sequence with the X's; so the map from the former spectral sequence to the latter is an isomorphism in columns n, $n+1$, and on the top two nonzero entries of column $n+2$. We will get an isomorphism on E^∞-terms in column $n+1$ if we can show that the single differential

$$\pi_{n+2}(A'_1, A_1) \to \pi_{n+1}(A_1, A)$$

vanishes in both spectral sequences.

The map $\pi_{n+2}(A'_1, A_1) \to \pi_{n+1}(A_1, A)$ factors through the map $\pi_{n+2}(A'_1, A_1) \to \pi_{n+1}(A_1)$, which is zero because $\pi_{n+1}(A_1) \to \pi_{n+1}(A'_1)$ is injective (see above). The same argument shows that $\pi_{n+2}(X'_1, X_1) \to \pi_{n+1}(X_1, X)$ is zero. This completes the proof. □

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