Solution for Quiz on 2/13

1. Prove that if $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a basis of V and $\mathbf{w}=\mathbf{a}+2\mathbf{b}+3\mathbf{c}$ that $V=\mathrm{Span}\{\mathbf{w},\mathbf{b},\mathbf{c}\}$. (See the second part of the proof of 2.3.2.2.)

Let $\mathbf{0}$ be the zero vector in V, and let s_1, s_2, s_3 be scalars such that
$\mathbf{0} = s_1\mathbf{w} + s_2\mathbf{b} + s_3\mathbf{c}$.
Substitute for $\mathbf{w}$ and combine like terms:
$\mathbf{0} = s_1\mathbf{a} + (2s_1+s_2)\mathbf{b} + (3s_1+s_3)\mathbf{c}$.
Since $\mathbf{a},\mathbf{b},\mathbf{c}$ are known to be linearly independent, we have
$s_1 = (2s_1+s_2) = (3s_1+s_3) = 0$.
It directly follows from this that
$s_1 = s_2 = s_3 = 0$.
Thus $\{\mathbf{w},\mathbf{b},\mathbf{c}\}$ are linearly independent. Since V is three-dimensional, $\mathrm{Span}\{\mathbf{w},\mathbf{b},\mathbf{c}\}=\mathbf{V}$.

2a. Define the rank of a matrix A.
The rank of a matrix is the dimension of its rowspace (or, equivalently, its columnspace); that is, it is the maximum number of linearly independent rows (or columns); that is, it is the number or nonzero rows (columns) in its row- (column-) echelon form.

2b. Calculate: $\mathrm{rank}\begin{pmatrix} e^2 & \pi^2 & 9 \\ e & \pi & 3 \\ 2e & 2\pi & 6 \end{pmatrix}$.

An (unreduced) row-echelon form of this matrix is: $\begin{pmatrix} e & \pi & 3 \\ 0 & \pi^2-e\pi & 9-3e \\ 0 & 0 & 0 \end{pmatrix}$

There are two nonzero rows, so the answer is 2.

3. Prove or disprove that $\mathbb{R}^2$ with
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$,
$c\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c\cdot x_1 \\ x_2 \end{pmatrix}$,
where $x_1, x_2, y_1, y_2, c \in \mathbb{R}$, is a vector space.

It is not a vector space. **Proof:** Call the structure in question V, and assume toward contradiction that it *is* a vector space. The vector addition rule for V is the same as the usual rule for $\mathbb{R}^2$. Therefore the zero vector in V is the same as usual, namely $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
For $x_1, x_2 \in \mathbb{R}$ with $x_2 \neq 0$, lemma 1.17 gives
$0\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
whereas by hypothesis
$0\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$.
If both equations are true, then $x_2 = 0$, contrary to assumption—CONTRADICTION. Thus V is not a vector space.