Solution for Quiz on 2/27

1. Refer to Three.I.1.16. Come up with two more isomorphisms between P_2 (polynomials of degree less than or equal to two) and $\mathbb{R}^3$. Do not use the ones from Example 1.2 or the solutions to Three.I.1.16. Make sure for yourself that your maps are isomorphisms (but for this problem you don't have to show your work or justify these maps are isomorphisms.)

Some possibilities:

$$f(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_1 \\ a_2 \\ a_0 \end{pmatrix}$$

$$g(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_0 + a_1 \\ a_0 + a_1 + a_2 \end{pmatrix}$$

$$h(a_0 + a_1x + a_2x^2) = \begin{pmatrix} -a_0 \\ -a_1 \\ -a_2 \end{pmatrix}$$

2. Produce an isomorphism between P_3 (polynomials of degree less than or equal to three) and the vector space of two by two real matrices ($M_{2\times 2}$). Show directly and in detail that your map is an isomorphism.

One of many possibilities:

$$f(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}.$$

Clearly this rule gives a well-defined set map.

f **is one-to-one:** Suppose the polynomials

$a_0 + a_1x + a_2x^2 + a_3x^3, b_0 + b_1x + b_2x^2 + b_3x^3 \in P_3$

are such that

$f(a_0 + a_1x + a_2x^2 + a_3x^3) = f(b_0 + b_1x + b_2x^2 + b_3x^3)$.

Applying the rule for f gives $\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} = \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix}$.

Since equal matrices have corresponding entries equal, we have $a_i = b_i$ for $i = 0, 1, 2, 3$. Thus in P_3,

$a_0 + a_1x + a_2x^2 + a_3x^3 = b_0 + b_1x + b_2x^2 + b_3x^3$.

Thus f is one-to-one.

f **is onto:** For any $\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} \in M_{2\times 2}$ the polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is in its preimage, as stated in the rule for f. Thus f is onto.

f **is linear:** Given polynomials

$A = a_0 + a_1x + a_2x^2 + a_3x^3, B = b_0 + b_1x + b_2x^2 + b_3x^3$

and scalar $r \in \mathbb{R}$ we have

$f(rA + B) = f(r(a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3))$

which, using vector scaling and addition in P_3, is

$= f((ra_0 + b_0) + (ra_1 + b_1)x + (ra_2 + b_2)x^2 + (ra_3 + b_3)x^3)$

which, using the rule for f, is

$$= \begin{pmatrix} ra_0 + b_0 & ra_1 + b_1 \\ ra_2 + b_2 & ra_3 + b_3 \end{pmatrix}$$

which, using vector scaling and addition in $M_{2\times 2}$, is

$$= r \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} + \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix}$$

which, using the rule for f, is

$$= rf(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + f(b_0 + b_1 x + b_2 x^2 + b_3 x^3)$$

$$= rf(A) + f(B).$$

Thus f is linear.