AN ALGEBRAIC MODEL FOR RATIONAL S^1-EQUIVARIANT STABLE HOMOTOPY THEORY

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ABSTRACT. Greenlees defined an abelian category $\mathcal{A}$ whose derived category is equivalent to the rational S^1-equivariant stable homotopy category whose objects represent rational S^1-equivariant cohomology theories. We show that in fact the model category of differential graded objects in $\mathcal{A}$ models the whole rational S^1-equivariant stable homotopy theory. That is, we show that there is a Quillen equivalence between $dg\mathcal{A}$ and the model category of rational S^1-equivariant spectra, before the quasi-isomorphisms or stable equivalences have been inverted. This implies that all of the higher order structures such as mapping spaces, function spectra and homotopy (co)limits are reflected in the algebraic model. The construction of this equivalence involves calculations with Massey products. In an appendix we show that Toda brackets, and hence also Massey products, are determined by the derived category.

1. INTRODUCTION

In [5], Greenlees defined an abelian category $\mathcal{A}$ and showed that its derived category is equivalent to the rational $\mathbb{T}$-equivariant stable homotopy category where $\mathbb{T}$ is the circle group. We strengthen this result by showing that this derived equivalence can be lifted to an equivalence on the underlying model categories, before the quasi-isomorphisms or stable equivalences have been inverted.

Theorem 1.1. *The model category of rational $\mathbb{T}$-equivariant spectra is Quillen equivalent to the model category of differential graded objects in $\mathcal{A}$.*

The definition of $\mathcal{A}$ and the model category on differential graded objects in $\mathcal{A}$, $dg\mathcal{A}$, are recalled in Section 2.

As mentioned in [5, 16.1], one of the applications of this stronger equivalence is that the algebraic models for the smash product and function spectra given in [5, Part IV] are natural. This also shows that the higher order structures such as mapping spaces and homotopy (co)limits are captured by the algebraic model [2, 3]. Since $\mathcal{A}$ has injective dimension one, calculations of these higher order structures are quite practical, as demonstrated here and in [5].

[5, Part III] discusses many motivations for studying rational $\mathbb{T}$-equivariant spectra, including its connection to algebraic K-theory, equivariant topological K-theory and Tate cohomology. This algebraic model has also been used to define a model for rational $\mathbb{T}$-equivariant elliptic cohomology [6]. Another reason for studying rational $\mathbb{T}$-equivariant spectra is that the circle is the simplest infinite compact Lie group. The arguments used in [5, 5.6.1] to show that the derived category, $\mathcal{D}(\mathcal{A})$, is equivalent to the homotopy category of rational $\mathbb{T}$-equivariant spectra, Ho($\mathbb{T}$-spectra), rely heavily on the fact that the circle is rank one. The general approach here though, using Quillen model categories and Morita equivalences, does apply to compact Lie groups of higher rank. Algebraic models of rational T^r-equivariant spectra, for tori of any rank, are considered in [7].

The main new ingredient here is a Morita equivalence for stable model categories considered in [22], see Section 3. In general, this Morita equivalence models a stable model category with a set of small generators by modules over a ring spectrum (with many objects). For rational stable model categories, this can be simplified further as (unbounded) modules over a rational *differential graded ring with many objects*; see Definition 3.1 or [23, 1.3]. The following statement is proved as Corollary 3.9 below. Definition 3.5 recalls the notion of a Quillen equivalence.

Theorem 1.2. *The model category of rational $\mathbb{T}$-equivariant spectra is Quillen equivalent to the model category of (right) modules over a rational differential graded ring $\mathcal{E}(\mathcal{B}_t)$ with one object for each closed subgroup in $\mathbb{T}$.*

$$\mathbb{T}\text{-}spectra \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$$

This gives an algebraic model for rational $\mathbb{T}$-equivariant spectra, but $\mathcal{E}(\mathcal{B}_t)$ is large and difficult to make explicit. Hence this algebraic model is not very practical. Instead, it is useful as a stepping stone to create a Quillen equivalence between $\mathbb{T}$-spectra and differential graded objects in $\mathcal{A}$. As another stepping stone we apply this same process to $dg\mathcal{A}$. The model category of differential graded objects in $\mathcal{A}$ is Quillen equivalent to (right) modules over a rational differential graded ring $\mathcal{E}(\mathcal{B}_a)$ with one object for each closed subgroup in $\mathbb{T}$; see Corollary 3.7 below. We then show that these two differential graded rings (with many objects) $\mathcal{E}(\mathcal{B}_t)$ and $\mathcal{E}(\mathcal{B}_a)$ are quasi-isomorphic. Since quasi-isomorphisms of differential graded rings (with many objects) induce Quillen equivalences of the associated categories of modules by [21, 4.3], see also [22, A.1.1] or [23, A.1], Theorem 1.1 follows given a zig-zag of quasi-isomorphisms between $\mathcal{E}(\mathcal{B}_a)$ and $\mathcal{E}(\mathcal{B}_t)$. This is carried out in Sections 4, 5 and 6.

Theorem 1.3. *The model category of differential graded objects in $\mathcal{A}$ is Quillen equivalent to modules over the differential graded ring $\mathcal{E}(\mathcal{B}_a)$. Also, there is a zig-zag of quasi-isomorphisms between $\mathcal{E}(\mathcal{B}_a)$ and $\mathcal{E}(\mathcal{B}_t)$ which induces a zig-zag of Quillen equivalences between the associated model categories. Hence we have a chain of Quillen equivalences*

$$\mathbb{T}\text{-}spectra \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t) \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_a) \simeq_Q dg\mathcal{A}.$$

In more detail, in Section 4 we explicitly describe $\mathcal{E}(\mathcal{B}_a)$ and a quasi-isomorphic sub-ring $\mathcal{E}_a$. In Section 5, we modify $\mathcal{E}(\mathcal{B}_t)$ to define a more amenable ring $\mathcal{E}_t$. Then in Section 6 we construct an intermediary differential graded ring $\mathcal{S}$ and two quasi-isomorphisms $\mathcal{E}_a \xleftarrow{\eta} \mathcal{S} \xrightarrow{\varphi} \mathcal{E}_t$. This construction involves Massey products for differential graded rings with many objects which are defined in Definition 4.8.

In particular, the proof of Theorem 1.3 uses the fact that the higher order products for $\mathcal{E}_a$ and $\mathcal{E}_t$ agree. These products agree because there are Quillen equivalences

$$\mathcal{M}od\text{-}\mathcal{E}_a \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_a) \simeq_Q dg\mathcal{A}$$

and

$$\mathcal{M}od\text{-}\mathcal{E}_t \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t) \simeq_Q \mathbb{T}\text{-}spectra$$

and $\mathrm{Ho}(\mathbb{T}\text{-}spectra)$ and $\mathcal{D}(\mathcal{A})$ are equivalent as triangulated categories by [5, 5.6.1]. This invariance of higher order products does not seem to appear in the literature, so we include it in an appendix on Toda brackets. A related statement, that quasi-isomorphisms preserve Massey products, appears in [18, 1.5]. Specifically, we prove the following as Theorem A.3.

Theorem 1.4. *If $\varphi \colon \mathcal{T} \longrightarrow \mathcal{T}'$ is an exact equivalence of triangulated categories then the Toda brackets for $\mathcal{T}$ and $\mathcal{T}'$ agree.*

2. Algebraic Model

In this section we recall Greenlees' algebraic model for rational $\mathbb{T}$-equivariant spectra; for more detail see [5, Chapters 4, 5; Appendix E.1]. We then show that this model is a $Ch_{\mathbb{Q}}$-model category; see Definition 2.4. Finally we recall certain objects called the algebraic basic cells.

First we recall some preliminary definitions.

Definition 2.1. [5, 4.5, 4.6, 5.2] Let $\mathcal{F}$ be the set of finite subgroups of $\mathbb{T}$. Let $\mathcal{O}_{\mathcal{F}}$ be the ring of operations, $\Pi_{H \in \mathcal{F}} \mathbb{Q}[c_H]$ with c_H in degree -2. Let e_H be the idempotent which is projection onto the factor corresponding to H. Let c be the *total Chern class* with $e_H c = c_H$. Let $\mathcal{E}$ be the multiplicative set of all *Euler classes*, $\{c^v \mid v \colon \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0}$ with finite support$\}$, where $e_H c^v = c_H^{v(H)}$. Let $t_*^{\mathcal{F}} = \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. As a vector space $(t_*^{\mathcal{F}})_{2n}$ is $\Pi_{H \in \mathcal{F}} \mathbb{Q}$ for $n \leq 0$ and is $\bigoplus_{H \in \mathcal{F}} \mathbb{Q}$ for $n > 0$.

We now define the abelian category $\mathcal{A}$. The category of differential graded objects in $\mathcal{A}$ is then the standard algebraic model of rational $\mathbb{T}$-equivariant spectra.

Definition 2.2. [5, 5.4.1] The objects of $\mathcal{A}$ are maps $N \xrightarrow{\beta} t_*^{\mathcal{F}} \otimes V$ of $\mathcal{O}_{\mathcal{F}}$-modules with V a graded vector space such that $\mathcal{E}^{-1}\beta$ is an isomorphism. N is referred to as the *nub* and V is the *vertex*. Morphisms are commutative squares

$$
\begin{array}{ccc}
M & \xrightarrow{\theta} & N \\
\beta \downarrow & & \downarrow \beta' \\
t_*^{\mathcal{F}} \otimes U & \xrightarrow{1 \otimes \phi} & t_*^{\mathcal{F}} \otimes V.
\end{array}
$$

The condition that $\mathcal{E}^{-1}\beta$ is an isomorphism is equivalent to requiring that the kernel K and cokernel C of β are $\mathcal{F}$-finite torsion modules [5, 4.5.1, 4.5.2, 4.6.6]. That is, they are torsion modules with respect to the total Chern class and they decompose as direct sums, $K = \bigoplus_H e_H K$ and $C = \bigoplus_H e_H C$.

Greenlees, in [5, 5.6.1], shows that the derived category of differential graded objects in $\mathcal{A}$ is equivalent to the homotopy category of rational $\mathbb{T}$-equivariant spectra. Here we recall the model category structure on $dg\mathcal{A}$. The associated homotopy category is equivalent to $\mathcal{D}(\mathcal{A})$.

Proposition 2.3. [5, Appendix B] *The category $dg\mathcal{A}$ of differential graded objects in $\mathcal{A}$ with cofibrations the monomorphisms, weak equivalences the quasi-isomorphisms, and fibrations determined by the right lifting property is a model category.*

The model category $dg\mathcal{A}$ is also tensored, cotensored and enriched over rational chain complexes in a way that is compatible with the model category structure. We call such a model category a $Ch_{\mathbb{Q}}$-model category. This is the analogue of a simplicial model category [20, II.2] with simplicial sets replaced by rational chain complexes. See [9, Ch. 4] for a more general definition. Here $Ch_{\mathbb{Q}}$ is the category of unbounded rational chain complexes with the projective model category structure [9, 2.3.3] and the standard closed symmetric monoidal structure [9, 4.2.13]. For definitions of *tensor, cotensor* and *enriched* see [13, 1.2, 3.7].

Definition 2.4. A $Ch_{\mathbb{Q}}$-*model category* is a complete and cocomplete model category $\mathcal{C}$ which is tensored and enriched (denoted $\mathrm{Hom}_{\mathcal{C}}$) over the category $Ch_{\mathbb{Q}}$, has cotensors with finite

dimensional complexes in $Ch_{\mathbb{Q}}$ and satisfies the following compatibility axiom (CM):

(CM) For every cofibration $A \xrightarrow{i} B$ and every fibration $X \xrightarrow{p} Y$ in $\mathcal{C}$ the induced map

$$\operatorname{Hom}_{\mathcal{C}}(i,p)\colon \operatorname{Hom}_{\mathcal{C}}(B,X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A,X) \times_{\operatorname{Hom}_{\mathcal{C}}(A,Y)} \operatorname{Hom}_{\mathcal{C}}(B,Y)$$

is a fibration in $Ch_{\mathbb{Q}}$. If in addition one of the maps i or p is a weak equivalence, then $\operatorname{Hom}_{\mathcal{C}}(i,p)$ is a trivial fibration. We use the notation $K \otimes X$ and X^K to denote the tensors and cotensors for X in $\mathcal{C}$ and K a chain complex.

For example, $Ch_{\mathbb{Q}}$ is itself a $Ch_{\mathbb{Q}}$-model category [9, 4.2.13] as is the projective model category defined in [22, A.1.1] or [23, A.1] of unbounded differential graded modules over any rational differential graded ring (with many objects); see Definition 3.1.

Proposition 2.5. *The model category on $dg\mathcal{A}$ is a $Ch_{\mathbb{Q}}$-model category.*

Proof. For $K \in Ch_{\mathbb{Q}}$ and $A \in dg\mathcal{A}$ define $K \otimes A$ in $dg\mathcal{A}$ by

$$K \otimes A = \bigoplus_n \Sigma^n(K_n \otimes_{\mathbb{Q}} A)$$

with differential $d(k_n \otimes a) = dk_n \otimes a + (-1)^n k_n \otimes da$. Let $\mathbb{Q}[n]$ denote the rational chain complex with $\mathbb{Q}$ in degree n. Then for A, B in $dg\mathcal{A}$ define $\operatorname{Hom}_{\mathcal{C}}(A,B)$ in $Ch_{\mathbb{Q}}$ by $\operatorname{Hom}_{\mathcal{C}}(A,B)_n = \mathcal{A}(\mathbb{Q}[n] \otimes A, B)$ where $\mathcal{A}(-,-)$ is the rational vector space of maps of underlying graded objects in $\mathcal{A}$ (ignoring the differential). The differential for $\operatorname{Hom}_{\mathcal{C}}(A,B)$ is given by $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. For K a finite dimensional complex, define A^K as $\operatorname{Hom}_{Ch_{\mathbb{Q}}}(K, \mathbb{Q}[0]) \otimes A$. This cotensor is the right adjoint of the tensor $K \otimes -$. So we have the following isomorphisms (where defined)

$$Ch_{\mathbb{Q}}(K, \operatorname{Hom}_{\mathcal{C}}(A,B)) \cong dg\mathcal{A}(K \otimes A, B) \cong dg\mathcal{A}(A, B^K).$$

As in [20, II.2 SM7(b)] or [9, 4.2.1, 4.2.2], the compatibility axiom in Definition 2.4 has an equivalent adjoint form involving the tensor. For $f\colon K \longrightarrow L$ in $Ch_{\mathbb{Q}}$ and $g\colon A \longrightarrow B$ in $dg\mathcal{A}$, define $f \square g\colon K \otimes B \amalg_{K \otimes A} L \otimes A \longrightarrow L \otimes B$. Then Axiom (CM) from Definition 2.4 is equivalent to requiring that if f and g are cofibrations, then $f \square g$ is a cofibration which is trivial if f or g is. Since $Ch_{\mathbb{Q}}$ is a cofibrantly generated model category we only need to check this property when f is one of the generating cofibrations or generating trivial cofibrations [21, 3.5].

Let D^n be the acyclic rational chain complex with $\mathbb{Q}$ in degrees n and $n-1$. Then the generating cofibrations of $Ch_{\mathbb{Q}}$ are the inclusions $i_n\colon \mathbb{Q}[n-1] \longrightarrow D^n$ for n an integer and the generating trivial cofibrations are the maps $j_n\colon 0 \longrightarrow D^n$ for n an integer [9, 2.3.3]. The source of the map $i_n \otimes g$ is the cofiber of g and the target is $D^n \otimes B$. So for g a monomorphism, $i_n \square g$ is a monomorphism because it is g on one summand and id_B on the other. Since $D^n \otimes B$ is always acyclic and the cofiber of a quasi-isomorphism is acyclic, if g is a trivial cofibration, then $i_n \square g$ is a quasi-isomorphism. Also, $j_n \square g\colon D^n \otimes A \longrightarrow D^n \otimes B$ is a quasi-isomorphism between acyclic complexes for any monomorphism g. □

For the Morita equivalence discussed in the next section we need to define the algebraic cells which can be used to build any object in $dg\mathcal{A}$. The natural building blocks for $\mathbb{T}$-equivariant spectra are the G-cells $\mathbb{T}_+ \wedge_H S^n$. Rationally, these can be simplified by using idempotents in the Burnside ring for a finite group, $A(H) = [S^0, S^0]^H \cong \Pi_{K \leq H} \mathbb{Q}$. Following Greenlees, we suppress the rationalization in our notation for spectra. Let e_K be the idempotent which is projection onto the Kth factor.

Definition 2.6. [5, 2.1.2] The *geometric basic cells* are the rational $\mathbb{T}$ spectra $\sigma_{\mathbb{T}}^0 = S^0$ and $\sigma_H^0 = \mathbb{T}_+ \wedge_H e_H S^0$ for H a finite subgroup of $\mathbb{T}$.

Via the equivalences between $\mathcal{D}(\mathcal{A})$ and rational $\mathbb{T}$-equivariant spectra, these geometric basic cells correspond to the following algebraic basic cells.

Definition 2.7. [5, 5.8.1] For H a finite subgroup, let $\mathbb{Q}(H)$ denote the $\mathbb{Q}[c_H]$-module $\mathbb{Q}$ regarded as an $\mathcal{O}_{\mathcal{F}}$-module. Then the *algebraic basic cells* are the objects $L_H = (\Sigma \mathbb{Q}(H) \longrightarrow 0)$ and $L_{\mathbb{T}} = (\mathcal{O}_{\mathcal{F}} \longrightarrow t^{\mathcal{F}}_*)$ in $\mathcal{A}$.

We need fibrant replacements of these algebraic basic cells which are inclusions into their respective injective hulls. The injective dimension of $\mathcal{A}$ is one, so the injective hulls are simple to construct. First we define the basic injective objects.

Definition 2.8. [5, 2.4.3, 3.4.1, 5.2.1, 5.5.1] Let $\mathbb{I}(H) = \Sigma^{-2}\mathbb{Q}[c_H, c_H^{-1}]/\mathbb{Q}[c_H]$. Let $\mathbb{I} = \bigoplus_H \mathbb{I}_H \cong \Sigma^{-2} t^{\mathcal{F}}_*/\mathcal{O}_{\mathcal{F}}$. The underlying vector spaces are $\mathbb{I}(H)_{2n} = \mathbb{Q}$ for $n \geq 0$ and $\mathbb{I}_{2n} = \bigoplus_H \mathbb{Q}$ for $n \geq 0$. Then $(\mathbb{I} \longrightarrow 0)$, $(\mathbb{I}(H) \longrightarrow 0)$, and $(t^{\mathcal{F}}_* \otimes V \longrightarrow t^{\mathcal{F}}_* \otimes V)$ for any graded vector space V are injective objects in $\mathcal{A}$.

The construction of fibrant approximations in [5, Appendix B] produces injective hulls for L_H and $L_{\mathbb{T}}$ which we denote by I_H and $I_{\mathbb{T}}$ respectively. Let $P(I) = I \ltimes \Sigma^{-1}I$ be the acyclic path object in $dg\mathcal{A}$ with underlying graded object $I \oplus \Sigma^{-1}I$. For a finite subgroup H of $\mathbb{T}$, let $\Sigma\mathbb{I}(H) \ltimes \Sigma^2\mathbb{I}(H)$ be the pullback in the following square.

$$
\begin{array}{ccc}
\Sigma\mathbb{I}(H) \ltimes \Sigma^2\mathbb{I}(H) & \longrightarrow & P(\Sigma^3\mathbb{I}(H)) \\
\downarrow & & \downarrow \\
\Sigma\mathbb{I}(H) & \longrightarrow & \Sigma^3\mathbb{I}(H)
\end{array}
$$

The bottom map is zero in degree one and an isomorphism in all higher degrees. Let I_H denote the object $(\Sigma\mathbb{I}(H) \ltimes \Sigma^2\mathbb{I}(H) \longrightarrow 0)$ in $dg\mathcal{A}$. The inclusion $L_H \longrightarrow I_H$ is a quasi-isomorphism and I_H is fibrant since it is built from standard injectives.

For $\mathbb{T}$ itself, let $t^{\mathcal{F}}_* \ltimes \Sigma\mathbb{I}$ be the pullback in the following square

$$
\begin{array}{ccc}
t^{\mathcal{F}}_* \ltimes \Sigma\mathbb{I} & \longrightarrow & P(\Sigma^2\mathbb{I}) \\
\downarrow & & \downarrow \\
t^{\mathcal{F}}_* & \longrightarrow & \Sigma^2\mathbb{I}
\end{array}
$$

The bottom map is zero in degrees $n \leq 0$ and an isomorphism in all positive degrees. Let $I_{\mathbb{T}}$ denote the object $(t^{\mathcal{F}}_* \ltimes \Sigma\mathbb{I} \longrightarrow t^{\mathcal{F}}_*)$ in $dg\mathcal{A}$. The inclusion $L_{\mathbb{T}} \longrightarrow I_{\mathbb{T}}$ is a quasi-isomorphism and $I_{\mathbb{T}}$ is fibrant since it is built from standard injectives.

Proposition 2.9. *I_H and $I_{\mathbb{T}}$ are fibrant replacements of L_H and $L_{\mathbb{T}}$ respectively.*

3. The Morita equivalence

Gabriel [4] proved a Morita theorem which shows that any cocomplete abelian category with a set of finitely generated projective generators is a category of $\mathcal{E}$-modules for some ring $\mathcal{E}$ (with many objects). See Definition 3.1 below. The ring $\mathcal{E}$ in question may be taken to be the endomorphism ring of the projective generators; this is naturally a ring with one object for each generator. An enriched variant of this Morita equivalence was considered in [22]. Here we consider another enriched variant which shows that a model category with a set of small generators which is compatibly enriched over chain complexes is Quillen equivalent to a category of differential graded modules over a differential graded ring (with many objects).

Definition 3.1. A *ring with objects* $\mathcal{G}$ is an *Ab*-category with object set $\mathcal{G}$. This is also a category enriched over abelian groups, [13, 1.2] or an *Ab*-module, [9, 4.1.6]. A ring is then an *Ab*-category with one object. Similarly, $\mathcal{R}$, a *rational differential graded ring with objects* $\mathcal{G}$,

is a category enriched over $Ch_{\mathbb{Q}}$, or a $Ch_{\mathbb{Q}}$-category or $Ch_{\mathbb{Q}}$-module. A (right) $\mathcal{R}$-module is a $Ch_{\mathbb{Q}}$-functor $M\colon \mathcal{R}^{\mathrm{op}} \longrightarrow Ch_{\mathbb{Q}}$.

Given $\mathcal{R}$, a rational differential graded ring with objects $\mathcal{G}$, and objects G, G', G'' in $\mathcal{G}$ there is a morphism chain complex $\mathrm{Hom}_{\mathcal{R}}(G, G') \in Ch_{\mathbb{Q}}$ and coherently associative and unital composition maps

$$\mathrm{Hom}_{\mathcal{R}}(G', G'') \otimes \mathrm{Hom}_{\mathcal{R}}(G, G') \longrightarrow \mathrm{Hom}_{\mathcal{R}}(G, G'').$$

An $\mathcal{R}$-module M is then determined by chain complexes $M(G)$ for each object $G \in \mathcal{G}$ and coherently associative and unital action maps

$$M(G') \otimes \mathrm{Hom}_{\mathcal{R}}(G, G') \longrightarrow M(G).$$

For each object G in $\mathcal{G}$ there is a free $\mathcal{R}$-module, $F_G^{\mathcal{R}}$, defined by $F_G^{\mathcal{R}}(G') = \mathcal{R}(G', G)$. The projective model structure on the category of $\mathcal{R}$-modules, $\mathcal{M}od\text{-}\mathcal{R}$, is established in [22, A.1.1] or [23, A.1] using [21, 2.3]. The weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms. With the standard action of $Ch_{\mathbb{Q}}$, $\mathcal{M}od\text{-}\mathcal{R}$ is a $Ch_{\mathbb{Q}}$-model category.

Definition 3.2. Let $\mathcal{G}$ be a set of objects in a $Ch_{\mathbb{Q}}$-model category $\mathcal{C}$. We denote by $\mathcal{E}(\mathcal{G})$ the full $Ch_{\mathbb{Q}}$-subcategory of $\mathcal{C}$ with objects $\mathcal{G}$, i.e., $\mathcal{E}(\mathcal{G})(G, G') = \mathrm{Hom}_{\mathcal{C}}(G, G')$ is a differential graded ring with many objects, one for each element of $\mathcal{G}$. We let

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, -) \,:\, \mathcal{C} \longrightarrow \mathcal{M}od\text{-}\mathcal{E}(\mathcal{G})$$

denote the functor given by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, Y)(G) = \mathrm{Hom}_{\mathcal{C}}(G, Y)$.

Note that, if $\mathcal{G} = \{G\}$ has a single element, then $\mathcal{E}(\mathcal{G})$ is determined by the single differential graded ring, $\mathrm{End}(G) = \mathrm{Hom}_{\mathcal{C}}(G, G)$.

For our enriched variant of Gabriel's theorem it is crucial that a $Ch_{\mathbb{Q}}$-model category is a *stable model category*, that is, a pointed model category where the suspension functor is a self-equivalence on the homotopy category [22]. The homotopy category of a stable model category is triangulated [9, 7.1.1, 7.1.6], [25]. The suspension and loop functors are inverse equivalences which induce the shift functor. The cofiber and fiber sequences agree up to a sign and induce the triangles.

Proposition 3.3. *If $\mathcal{C}$ is a $Ch_{\mathbb{Q}}$-model category then $\mathcal{C}$ is a stable model category. If X is a cofibrant object in $\mathcal{C}$ and Y is a fibrant object in $\mathcal{C}$, then there is a natural isomorphism of graded abelian groups $H_* \mathrm{Hom}_{\mathcal{C}}(X, Y) \cong [X, Y]_*^{\mathrm{Ho}(\mathcal{C})}$.*

Proof. First, $\mathcal{C}$ is pointed since $Ch_{\mathbb{Q}}$ is pointed [9, 4.2.19]. For a cofibrant object X in $\mathcal{C}$, a cylinder object [20, I.1] is given by $(D^1 \oplus \mathbb{Q}[0]) \otimes X$. It is a cylinder object because $X \amalg X \cong (\mathbb{Q}[0] \oplus \mathbb{Q}[0]) \otimes X \xrightarrow{i} (D^1 \oplus \mathbb{Q}[0]) \otimes X$ is a cofibration, by an adjoint form of the compatibility axiom (CM), see [20, II.2, SM7(b)], and $(D^1 \oplus \mathbb{Q}[0]) \otimes X \longrightarrow \mathbb{Q}[0] \otimes X \cong X$ is a quasi-isomorphism. Hence the cofiber of i, $\mathbb{Q}[1] \otimes X$, represents ΣX. But $\mathbb{Q}[1]$ is invertible with inverse $\mathbb{Q}[-1]$. Since the action of $Ch_{\mathbb{Q}}$ on $\mathcal{C}$ is associative up to coherent isomorphism, this shows that Σ is a self-equivalence on $\mathrm{Ho}(\mathcal{C})$. Thus $\mathcal{C}$ is a stable model category.

Since a cylinder object in $Ch_{\mathbb{Q}}$ is also given by tensoring with $D^1 \oplus \mathbb{Q}[0]$, we have the following natural isomorphisms for a cofibrant object X in $\mathcal{C}$ and a fibrant object Y in $\mathcal{C}$.

$$H_n \mathrm{Hom}_{\mathcal{C}}(X, Y) \cong [\mathbb{Q}[n], \mathrm{Hom}_{\mathcal{C}}(X, Y)]^{\mathrm{Ho}(Ch_{\mathbb{Q}})} \cong [\mathbb{Q}[n] \otimes X, Y]^{\mathrm{Ho}(\mathcal{C})} \cong [\Sigma^n X, Y]^{\mathrm{Ho}(\mathcal{C})}$$

$\square$

We also need the following definitions from triangulated categories, see [11, 1.1].

Definition 3.4. An object X in a stable model category $\mathcal{C}$ is *small* if in $\mathrm{Ho}(\mathcal{C})$ the natural map $\bigoplus_i [X, Y_i] \longrightarrow [X, \mathrm{II}_i Y_i]$ is an isomorphism for any set of objects $\{Y_i\}$. A subcategory of a triangulated category is *localizing* if it is closed under cofiber sequences, retracts and coproducts. A set of objects $\mathcal{G}$ in $\mathcal{C}$ is a set of (weak) *generators* if the only localizing subcategory of $\mathrm{Ho}(\mathcal{C})$ which contains $\mathcal{G}$ is $\mathrm{Ho}(\mathcal{C})$.

The following enriched variant of Gabriel's theorem is stated rationally, but holds integrally as well with $Ch_{\mathbb{Q}}$ replaced by $Ch_{\mathbb{Z}}$. A similar statement, on the derived category level, can be found in [12]. Here though we consider the underlying model categories, before inverting the quasi-isomorphisms, to construct a Quillen equivalence.

Definition 3.5. A pair of adjoint functors between model categories is a *Quillen adjoint pair* if the right adjoint preserves fibrations and trivial fibrations. A Quillen adjoint pair induces adjoint total derived functors between the homotopy categories [20, I.4]. A Quillen adjoint pair is a *Quillen equivalence* if the total derived functors are adjoint equivalences of the homotopy categories. This is equivalent to the usual definition by [9, 1.3.13].

Theorem 3.6. *Let $\mathcal{C}$ be a $Ch_{\mathbb{Q}}$-model category and $\mathcal{G}$ a set of small, cofibrant and fibrant generators. The functor*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, -) \, : \, \mathcal{C} \longrightarrow \mathcal{M}od\text{-}\mathcal{E}(\mathcal{G})$$

is the right adjoint of a Quillen equivalence. The left adjoint is denoted $- \otimes_{\mathcal{E}(\mathcal{G})} \mathcal{G}$.

Proof. The functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, -)$ preserves fibrations and trivial fibrations by the compatibility axiom (CM) of Definition 2.4, since all objects of $\mathcal{G}$ are cofibrant. So together with its left adjoint it forms a Quillen pair. Denote the associated total derived functors on the homotopy categories by $\mathrm{RHom}_{\mathcal{C}}(\mathcal{G}, -)$ and $- \otimes_{\mathcal{E}(\mathcal{G})}^L \mathcal{G}$. Since $\mathrm{Ho}(\mathcal{C})$ is a triangulated category both total derived functors preserve shifts and triangles; they are exact functors of triangulated categories by [20, I.4]. Since $- \otimes_{\mathcal{E}(\mathcal{G})}^L \mathcal{G}$ is a left adjoint it commutes with colimits. The right adjoint also commutes with coproducts since each object of $\mathcal{G}$ is small. This follows as in [22, 3.10.3(ii)].

We consider the full subcategories of those $M \in \mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}(\mathcal{G}))$ and $X \in \mathrm{Ho}(\mathcal{C})$ respectively for which the unit of the adjunction

$$\eta \, : \, M \longrightarrow \mathrm{RHom}_{\mathcal{C}}(\mathcal{G}, M \otimes_{\mathcal{E}(\mathcal{G})}^L \mathcal{G})$$

or the counit of the adjunction

$$\nu \, : \mathrm{RHom}_{\mathcal{C}}(\mathcal{G}, X) \otimes_{\mathcal{E}(\mathcal{G})}^L X \longrightarrow X$$

are isomorphisms. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. For every $G \in \mathcal{G}$ the $\mathcal{E}(\mathcal{G})$-module $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, G)$ is isomorphic to the free module $F_G^{\mathcal{E}(\mathcal{G})}$ by inspection and $F_G^{\mathcal{E}(\mathcal{G})} \otimes_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ is isomorphic to G since they represent the same functor on $\mathcal{C}$. So the map η is an isomorphism for every free module, and the map ν is an isomorphism for every object of $\mathcal{G}$. Since the free modules $F_G^{\mathcal{E}(\mathcal{G})}$ generate the homotopy category of $\mathcal{E}(\mathcal{G})$-modules and the objects of $\mathcal{G}$ generate $\mathcal{C}$, the localizing subcategories are the whole categories. So the derived functors are inverse equivalences of the homotopy categories. $\square$

Since $dg\mathcal{A}$ is a $Ch_{\mathbb{Q}}$-model category by Proposition 2.5, we can apply Theorem 3.6. Triangulated equivalences preserve generators so we identify generators of $dg\mathcal{A}$ via the triangulated equivalence $\pi^{\mathcal{A}} \colon \mathrm{Ho}(\mathbb{T}\text{-spectra}) \longrightarrow \mathcal{D}(\mathcal{A})$ from the homotopy category of $\mathbb{T}$-spectra to the homotopy category of $dg\mathcal{A}$, or the derived category of $\mathcal{A}$, by [5, 5.6.1]. Note that composition of $\pi^{\mathcal{A}}$ with homology gives the functor $\pi_*^{\mathcal{A}}$ which appears in [5, 5.6.2]. The geometric basic cells $\mathcal{B}_t' = \{\sigma_H^0\}_{H \leq \mathbb{T}}$ for H any closed subgroup of $\mathbb{T}$ form a set of small generators for $\mathbb{T}$-spectra.

This set is labeled $\mathcal{B}'_t$ because we will soon consider its image $\mathcal{B}_t$ in another Quillen equivalent category. They detect stable equivalences because $\pi^H_*(X) = [\vee_{K \leq H} \sigma^0, X]^{\mathbb{T}}_*$ for H finite and $\pi^{\mathbb{T}}_*(X) = [\sigma^0_{\mathbb{T}}, X]^{\mathbb{T}}_*$ by [5, 2.1.5]. It follows by [22, 2.2.1] or part of [11, 2.3.2] that $\mathcal{B}'_t$ is a set of small generators. Hence, the set of images $\{\pi^{\mathcal{A}}(\sigma^0_H) = L_H\}_{H \leq \mathbb{T}}$ form a set of small generators for $dg\mathcal{A}$. Let $\mathcal{B}_a = \{I_H\}_{H \leq \mathbb{T}}$ be the set of their fibrant replacements. Then $\mathcal{B}_a$ is a set of small, cofibrant and fibrant generators. Let $\mathcal{E}(\mathcal{B}_a)$ be the endomorphism $Ch_{\mathbb{Q}}$-category as in Definition 3.2. In Section 4 we explicitly describe $\mathcal{E}(\mathcal{B}_a)$.

Corollary 3.7. *The model category $dg\mathcal{A}$ is Quillen equivalent to the model category of differential graded modules over $\mathcal{E}(\mathcal{B}_a)$.*

To prepare for constructing a Quillen equivalence between $dg\mathcal{A}$ and the model category of rational $\mathbb{T}$-equivariant spectra, $\mathbb{T}$-spectra, we show that there is an equivalence between $\mathbb{T}$-spectra and modules over a rational differential graded ring with many objects. A Morita equivalence similar to Theorem 3.6 in [22] considers Sp^{Σ}-model categories, or *spectral model categories*, model categories compatibly enriched over symmetric spectra. [22, 3.10.3] shows that any spectral model category with a set of small generators is equivalent to modules over a ring spectrum with many objects. In this paper we are interested in particular in *rational stable model categories*, stable model categories where $[X, Y]^{\text{Ho}(\mathcal{C})}$ is a rational vector space for any objects X and Y in $\mathcal{C}$. In this case the Gabriel equivalence produces a rational ring spectrum which can be replaced by a differential graded ring with many objects.

Theorem 3.8. [23, 1.3] *Let $\mathcal{C}$ be a spectral model category. If $\mathcal{C}$ is rational and has a set of small generators, then $\mathcal{C}$ is Quillen equivalent (via a chain of several Quillen equivalences) to the model category of modules over a rational differential graded ring with many objects.*

Before rationalization, the category of $\mathbb{T}$-equivariant spectra is a cellular, simplicial, left proper, stable model category [16, III.4.2]. In particular, a G-topological model category [16, III.1.14] is a topological model category which in turn is a simplicial model category via the singular and realization functors. Also, a compactly generated model category is a stricter notion than a cofibrantly generated model category. One can check that $\mathbb{T}$-equivariant spectra is in fact cellular, see [10, A.1] or [8, 14.1.1]. Since localizations preserve these properties by [8, 4.1.1], the model category of rational $\mathbb{T}$-equivariant spectra is also a cellular, simplicial, left proper, stable model category. Then Hovey in [10, 8.11,9.1] shows that $\mathbb{T}$-spectra is Quillen equivalent to a spectral model category, $\mathcal{C}'$. Since $\mathbb{T}$-spectra is also a simplicial, cofibrantly generated, proper, stable model category, [22, 3.9.2] also shows it is Quillen equivalent to a spectral model category. Since rational $\mathbb{T}$-equivariant spectra has a set of small generators, $\mathcal{B}'_t$, the geometric basic cells, $\mathcal{C}'$ is also rational and has a set of small generators because these properties are determined on the homotopy category level. Thus, by Theorem 3.8, $\mathcal{C}'$ is Quillen equivalent to $\mathcal{M}od\text{-}\mathcal{R}$ for some rational differential graded ring with many objects, $\mathcal{R}$. Let $\mathcal{B}_t$ be a set of cofibrant and fibrant replacements of the images of the basic cells in $\mathcal{M}od\text{-}\mathcal{R}$. Then $\mathcal{B}_t$ is a set of small generators in $\mathcal{M}od\text{-}\mathcal{R}$. Since $\mathcal{M}od\text{-}\mathcal{R}$ is a $Ch_{\mathbb{Q}}$-model category, Theorem 3.6 shows that $\mathcal{M}od\text{-}\mathcal{R}$ is Quillen equivalent to $\mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$ where $\mathcal{E}(\mathcal{B}_t)$ is the endomorphism ring on the generators $\mathcal{B}_t$.

Corollary 3.9. *$\mathbb{T}$-spectra is Quillen equivalent to $\mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$ which is the endomorphism ring on a set of small generators $\mathcal{B}_t$ in an intermediate model category $\mathcal{M}od\text{-}\mathcal{R}$.*

$$\mathbb{T}\text{-}spectra \simeq_Q \mathcal{M}od\text{-}\mathcal{R} \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$$

This does give an algebraic model for $\mathbb{T}$-spectra, but $\mathcal{E}(\mathcal{B}_t)$ is large and not explicit. This algebraic model is nonetheless useful because we can construct a zig-zag of quasi-isomorphisms between $\mathcal{E}(\mathcal{B}_t)$ and $\mathcal{E}(\mathcal{B}_a)$ which induce Quillen equivalences on the respective module categories.

4. The endomorphism ring for $dg\mathcal{A}$

In this section we first explicitly calculate $\mathcal{E}(\mathcal{B}_a)$, the endomorphism ring associated to $dg\mathcal{A}$ by Corollary 3.7. Then in Definition 4.6 we define a sub-ring $\mathcal{E}_a$ which is quasi-isomorphic to $\mathcal{E}(\mathcal{B}_a)$. In Section 6, we then use this simpler ring $\mathcal{E}_a$ to construct a zig-zag of Quillen equivalences between $dg\mathcal{A}$ and $\mathbb{T}$-spectra. For this construction, we need to understand the Massey products in $H_*\mathcal{E}_a$, so we end this section with a definition of Massey products for differential graded rings with many objects and a calculation of the products and Massey products in $H_*\mathcal{E}_a$. Since $\mathcal{E}_a$ and $\mathcal{E}(\mathcal{B}_a)$ are quasi-isomorphic this calculation also applies to $H_*\mathcal{E}(\mathcal{B}_a)$ by [18, 1.5].

First we calculate the underlying rational chain complexes of $\mathcal{E}(\mathcal{B}_a)$.

Proposition 4.1. *For any distinct finite subgroups H, K of $\mathbb{T}$ the rational chain complexes of homomorphisms are as follows:*

1. $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_H) = (\bigoplus_{n \leq 0} D^n) \oplus \mathbb{Q}[0] \oplus \mathbb{Q}[1]$,
2. $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_K) = 0$,
3. $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_\mathbb{T}) = (\bigoplus_{n \leq 0} D^{2n-1}) \oplus \mathbb{Q}[0]$,
4. $\mathrm{Hom}_{dg\mathcal{A}}(I_\mathbb{T}, I_H) = (\bigoplus_{n \geq 0} D^{2n+1}) \oplus (\bigoplus_{m \leq 0} D^m) \oplus \mathbb{Q}[1]$ *and*
5. $\mathrm{Hom}_{dg\mathcal{A}}(I_\mathbb{T}, I_\mathbb{T}) = (\bigoplus_{n \geq 0} \mathbb{Q}\mathcal{F}[2n+1]) \oplus (\bigoplus_{m \leq 0} (D^{2m} \otimes \mathbb{Q}\mathcal{F})) \oplus \mathbb{Q}[0]$ *where $\mathbb{Q}\mathcal{F}$ is the rational vector space with basis $\mathcal{F}$, the set of finite subgroups of $\mathbb{T}$.*

Proof. In the first three cases, since the vertex of the source is trivial the maps are determined by considering graded $\mathcal{O}_\mathcal{F}$-module maps of the nubs, denoted by $\mathcal{O}_{\mathcal{F}*}(-, -)$. As an underlying $\mathcal{O}_\mathcal{F}$-module, the nub of I_H is $\Sigma\mathbb{I}(H) \oplus \Sigma^2\mathbb{I}(H)$. As a graded vector space $\mathcal{O}_{\mathcal{F}*}(\mathbb{I}(H), \mathbb{I}(H)) = \bigoplus_{n \leq 0} \mathbb{Q}[2n]$. This determines the underlying graded vector space of $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_H)$. Let $\{m_{2n}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma\mathbb{I}(H))$, $\{m_{2n+1}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma^2\mathbb{I}(H))$, $\{l_{2n}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma^2\mathbb{I}(H), \Sigma^2\mathbb{I}(H))$ and $\{l_{2n-1}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma^2\mathbb{I}(H), \Sigma\mathbb{I}(H))$. When necessary, these bases and those below are decorated with an H. The differential is determined by $dm_{2n} = m_{2n-1}$, $dm_{2n+1} = 0$, $dl_{2n} = m_{2n-1}$ and $dl_{2n-1} = l_{2n-2} - m_{2n-2}$. Since $\mathcal{O}_{\mathcal{F}*}(\mathbb{I}(H), \mathbb{I}(K)) = 0$, it follows that $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_K) = 0$.

As an underlying $\mathcal{O}_\mathcal{F}$-module, the nub of $I_\mathbb{T}$ is $t_*^\mathcal{F} \oplus \Sigma\mathbb{I}$. As a graded vector space $\mathcal{O}_{\mathcal{F}*}(\mathbb{I}(H), \mathbb{I}) = \bigoplus_{n \leq 0} \mathbb{Q}[2n]$ and $\mathcal{O}_{\mathcal{F}*}(\mathbb{I}(H), t_*^\mathcal{F}) = 0$; this determines the underlying graded vector space of $\mathrm{Hom}_{dg\mathcal{A}}(I_H, I_\mathbb{T})$. Let $\{f_{2n}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma\mathbb{I})$ and $\{f_{2n-1}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma^2\mathbb{I}(H), \Sigma\mathbb{I})$. The differential is determined by $df_{2n-1} = -f_{2n-2}$ and $df_{2n} = 0$.

For the fourth case the vertex of the target is trivial, so again the maps are determined by the nubs. Also, note that $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, \mathbb{I}(H)) = \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}[2n]$. Let $\{g_{2n+1}\}_{n \in \mathbb{Z}}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, \Sigma\mathbb{I}(H))$, $\{g_{2n}\}_{n \in \mathbb{Z}}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, \Sigma^2\mathbb{I}(H))$, $\{h_{2n}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma\mathbb{I}(H))$ and $\{h_{2n+1}\}_{n \leq 0}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma^2\mathbb{I}(H))$. The differential is determined by $dg_{2n+1} = g_{2k}$, $dg_{2n} = 0$, $dh_{2n} = h_{2n-1} + g_{2n-1}$ and $dh_{2n+1} = -g_{2n}$.

The last case is the only one where we must consider the vertex. This restricts the maps from $t_*^\mathcal{F} \otimes \mathbb{Q}$ to itself to maps of the form $1 \otimes \phi$. Hence $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, t_*^\mathcal{F})$ contributes only $\mathbb{Q}[0]$ to $\mathrm{Hom}_{dg\mathcal{A}}(I_\mathbb{T}, I_\mathbb{T})$. Also, note that $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, \mathbb{I}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}\mathcal{F}[2n]$ and $\mathcal{O}_{\mathcal{F}*}(\mathbb{I}, \mathbb{I}) = \bigoplus_{n \leq 0} \mathbb{Q}\mathcal{F}[2n]$. Let $\{i_{2n+1}^H\}_{n \in \mathbb{Z}, H \in \mathcal{F}}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(t_*^\mathcal{F}, \Sigma\mathbb{I}(H))$, $\{j_{2n}^H\}_{n \leq 0, H \in \mathcal{F}}$ be a basis for $\mathcal{O}_{\mathcal{F}*}(\Sigma\mathbb{I}(H), \Sigma\mathbb{I}(H))$ and $\mathrm{id}_\mathbb{T}$ the identity map. The differential is determined by $di_{2n+1}^H = 0$, $dj_{2n}^H = i_{2n-1}^H$ and $d\,\mathrm{id}_\mathbb{T} = 0$. $\square$

As we note below in Proposition 4.4, for the rest of this paper we only need to understand the product structure on the (-1)-connected cover of $\mathcal{E}(\mathcal{B}_a)$. This greatly simplifies the structure, see Proposition 4.5. So only the non-negative dimensional products in the following proposition are needed and these products are displayed in Definition 4.6.

Proposition 4.2. *The following is a list of all of the non-trivial products, or compositions, of the bases chosen for $\mathcal{E}(\mathcal{B}_a)$ in Proposition 4.1.*

$$m_{2n}g_{2k+1} = g_{2n+2k+1} \qquad m_{2n+1}g_{2k+1} = g_{2n+2k+2} \qquad l_{2n}g_{2k} = g_{2n+2k} \qquad l_{2n-1}g_{2k} = g_{2n+2k-1}$$
$$f_{2n-1}g_{2k} = i_{2n+2k-1} \qquad f_{2n}g_{2k+1} = i_{2n+2k+1} \qquad l_{2n}h_{2k} = g_{2n+2k} \qquad l_{2n-1}h_{2k} = g_{2n+2k-1}$$
$$l_{2n}h_{2k+1} = h_{2n+2k+1} \qquad l_{2n-1}h_{2k+1} = h_{2n+2k} \qquad h_{2n}f_{2k} = m_{2n+2k} \qquad h_{2n}f_{2k-1} = l_{2n+2k-1}$$
$$h_{2n+1}f_{2k} = m_{2n+2k+1} \qquad h_{2n+1}f_{2k-1} = l_{2n+2k} \qquad f_{2n}h_{2k} = j_{2n+2k} \qquad f_{2n-1}h_{2k+1} = j_{2n+2k}$$
$$h_{2n}j_{2k} = h_{2n+2k} \qquad h_{2n+1}j_{2k} = h_{2n+2k+1} \qquad h_{2n}i_{2k+1} = g_{2n+2k+1} \qquad h_{2n+1}i_{2k+1} = g_{2n+2k+2}$$
$$f_{2n}m_{2k} = f_{2n+2k} \qquad f_{2n-1}m_{2k+1} = f_{2n+2k} \qquad f_{2n-1}l_{2k} = f_{2n+2k-1} \qquad f_{2n}l_{2k-1} = f_{2n+2k-1}$$
$$j_{2n}f_{2k} = f_{2n+2k} \qquad j_{2n}f_{2k-1} = f_{2n+2k-1} \qquad m_{2n}m_{2k} = m_{2n+2k} \qquad m_{2n+1}m_{2k} = m_{2n+2k+1}$$
$$m_{2n}l_{2k-1} = l_{2n+2k-1} \qquad m_{2n+1}l_{2k-1} = l_{2n+2k} \qquad l_{2n}m_{2k+1} = m_{2n+2k+1} \qquad l_{2n-1}m_{2k+1} = m_{2n+2k}$$
$$l_{2n}l_{2k} = l_{2n+2k} \qquad l_{2n-1}l_{2k} = l_{2n+2k-1} \qquad j_{2n}^{H}j_{2k}^{H} = j_{2n+2k}^{H} \qquad j_{2n}^{H}i_{2k+1}^{H} = i_{2n+2k+1}^{H}$$

Also, $x\,\mathrm{id}_{\mathbb{T}} = x$ where $x = i_{2n+1}, j_{2n}, \mathrm{id}_{\mathbb{T}}$ or f_n and $\mathrm{id}_{\mathbb{T}}\, y = y$ where $y = i_{2n+1}, j_{2n}, \mathrm{id}_{\mathbb{T}}, g_n$ or h_n.

Next we consider simpler sub-rings of $\mathcal{E}(\mathcal{B}_a)$.

Definition 4.3. The (-1)-*connected cover* $R\langle 0\rangle$ of a differential graded ring R is defined by $R\langle 0\rangle_n = 0$ for $n < 0$, $R\langle 0\rangle_n = R_n$ for $n > 0$ and $R\langle 0\rangle_0 = Z_0 R$ where $Z_0 R$ is the kernel of the differential $d\colon R_0 \longrightarrow R_{-1}$. Since the product of two cycles is a cycle, the product on R induces a product on $R\langle 0\rangle$. Moreover the inclusion $R\langle 0\rangle \longrightarrow R$ is a map of differential graded rings which induces an isomorphism in homology in non-negative degrees. This definition generalizes to a differential graded ring with many objects, $\mathcal{R}$, by applying this to each morphism complex $\mathcal{R}(o, o')$.

Since the homology of $\mathcal{E}(\mathcal{B}_a)$ is concentrated in non-negative degrees it is quasi-isomorphic to its (-1)-connected cover. By [21, 4.3], the module categories are Quillen equivalent.

Proposition 4.4. *The map $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle \longrightarrow \mathcal{E}(\mathcal{B}_a)$ is a quasi-isomorphism of differential graded rings with objects $\mathcal{B}_a$. Thus, there is a Quillen equivalence between the respective categories of modules.*

$$\mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_a)\langle 0\rangle \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_a)$$

Next we explicitly describe $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle$.

Proposition 4.5. *The rational chain complexes underlying the differential graded ring $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle$ are given by:*
1. *$\mathcal{E}(\mathcal{B}_a)\langle 0\rangle(I_H, I_H) = \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ with generators $\mathrm{id}_H = m_0 + l_0$ and m_1,*
2. *$\mathcal{E}(\mathcal{B}_a)\langle 0\rangle(I_H, I_K) = 0$,*
3. *$\mathcal{E}(\mathcal{B}_a)\langle 0\rangle(I_H, I_{\mathbb{T}}) = \mathbb{Q}[0]$ with generator f_0,*
4. *$\mathcal{E}(\mathcal{B}_a)\langle 0\rangle(I_{\mathbb{T}}, I_H) = (\bigoplus_{n\geq 0} D^{2n+1}) \oplus \mathbb{Q}[1]$ with generators g_n for degrees $n \geq 2$, $g_1 + h_1$ generating the cycles in degree 1 and h_1 and g_0 generating D^1 and*
5. *$\mathcal{E}(\mathcal{B}_a)\langle 0\rangle(I_{\mathbb{T}}, I_{\mathbb{T}}) = (\bigoplus_{n\geq 0} \mathbb{Q}\mathcal{F}[2n+1]) \oplus \mathbb{Q}[0]$ with generators i_{2n+1}^H and $\mathrm{id}_{\mathbb{T}}$ with $n \geq 0$.*

The products are determined by the products in $\mathcal{E}(\mathcal{B}_a)$.

We make one more modification to the ring $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle$ by considering the sub-ring $\mathcal{E}_a$.

Definition 4.6. Define the differential graded ring $\mathcal{E}_a$ with objects $\mathcal{B}_a$ as the sub-ring of $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle$ produced by deleting the D^1 generated by h_1 and g_0. Denote the element $g_1 + h_1$ in $\mathcal{E}(\mathcal{B}_a)\langle 0\rangle$ by $\widetilde{g}_1$ in $\mathcal{E}_a$. To ease notation in Section 6, the element $\widetilde{g}_1$ will be renamed g_1 to match the family of elements g_{2n+1}. Then the rational chain complexes underlying $\mathcal{E}_a$ are given by:
1. $\mathcal{E}_a(I_H, I_H) = \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ with generators $\mathrm{id}_H = m_0^H + l_0^H$ and m_1^H,
2. $\mathcal{E}_a(I_H, I_K) = 0$,

3. $\mathcal{E}_a(I_H, I_{\mathbb{T}}) = \mathbb{Q}[0]$ with generator f_0^H,
4. $\mathcal{E}_a(I_{\mathbb{T}}, I_H) = (\bigoplus_{n \geq 1} D^{2n+1}) \oplus \mathbb{Q}[1]$ with generators $\widetilde{g}_1^H$, g_n^H for $n \geq 2$, and
5. $\mathcal{E}_a(I_{\mathbb{T}}, I_{\mathbb{T}}) = (\bigoplus_{n \geq 0} \mathbb{Q}\mathcal{F}[2n+1]) \oplus \mathbb{Q}[0]$ with generators i_{2n+1}^H and $\mathrm{id}_{\mathbb{T}}$ with $n \geq 0$.

The non-trivial products of these chosen bases, except the obvious products with id_H and $\mathrm{id}_{\mathbb{T}}$ are:

1. $\widetilde{g}_1^H f_0^H = m_1^H$,
2. $f_0^H g_{2n+1}^H = i_{2n+1}^H$ for $n \geq 0$ and
3. $\widetilde{g}_1^H f_0^H g_{2n+1}^H = g_{2n+2}^H$ for $n \geq 0$.

When $n = 0$, the second line should be read as $f_0 \widetilde{g}_1^H = i_1^H$ and the third line should be read as $\widetilde{g}_1^H f_0^H \widetilde{g}_1^H = g_2^H$. The products in the third line can also be rewritten in terms of m_1^H or i_{2n+1}^H using the first two lines.

Another description of $\mathcal{E}_a$ is that it is the differential graded ring generated by the elements f_0^H, $\widetilde{g}_1^H$ and g_{2n+1}^H for $n > 0$ with relations generated by $f_0^H \widetilde{g}_1^H f_0^H = 0$, $g_{2n+1}^H f_0^H = 0$ for $n > 0$ and $x^H y^K = 0$ whenever $H \neq K$. These relations are equivalent to requiring $\mathcal{E}_a(I_H, I_{\mathbb{T}}) \cong \mathbb{Q}[0]$, $\mathcal{E}_a(I_H, I_H) \cong \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ and $\mathcal{E}_a(I_H, I_K) = 0$.

Since quasi-isomorphisms induce Quillen equivalences by [21, 4.3], we have the following statement.

Proposition 4.7. *The inclusion of differential graded rings $\mathcal{E}_a \longrightarrow \mathcal{E}(\mathcal{B}_a)\langle 0 \rangle$ which sends $\widetilde{g}_1$ to $g_1 + h_1$ is a quasi-isomorphism. Hence the composite $\mathcal{E}_a \longrightarrow \mathcal{E}(\mathcal{B}_a)\langle 0 \rangle \longrightarrow \mathcal{E}(\mathcal{B}_a)$ is also a quasi-isomorphism. Thus, there is a Quillen equivalence between the respective categories of modules.*

$$\mathcal{M}od\text{-}\mathcal{E}_a \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_a)$$

We now consider higher order products. Here the higher order products we consider are Massey products. We generalize the definition of Massey products for a differential graded algebra due to Massey [17] to a differential graded ring with many objects; see also [19, 8.3.2]. Since there is no real difference between the homological and cohomological versions we still refer to these as Massey products instead of the 'Yessam' products of Stasheff [24]. Notice that with this definition the indeterminacy may be smaller for a differential graded ring with many objects than for the associated differential graded matrix ring on only one object. To deal with signs here we define $\epsilon(a) = (-1)^{\deg a + 1} a$ instead of using the usual bar notation.

Definition 4.8. Let $\mathcal{R}$ be a differential graded ring with objects $\mathcal{G} = \{G_i\}_{i \in I}$. Suppose that $\gamma_1, \ldots, \gamma_n$ are classes in $H_* \mathcal{R}$ with $\gamma_i \in H_{p_i} \mathcal{R}(G_i, G_{i-1})$. A *defining system*, associated to $\langle \gamma_1, \ldots, \gamma_n \rangle$, is a set of elements $a_{i,j}$ for $1 \leq i \leq j \leq n$ with $(i, j) \neq (1, n)$ such that

1. $a_{i,j} \in \mathcal{R}(G_j, G_{i-1})_{p_i + \cdots + p_j + j - i}$,
2. $a_{i,i}$ is a cycle representative of γ_i and
3. $d(a_{i,j}) = \Sigma_{r=i}^{j-1} \epsilon(a_{i,r}) a_{r+1,j}$ with $\epsilon(a) = (-1)^{\deg a + 1} a$.

To each defining system associate the cycle

$$\Sigma_{r=1}^{n-1} \epsilon(a_{1,r}) a_{r+1,n} \in \mathcal{R}(G_n, G_0)_{p_1 + \ldots p_n + n - 2}$$

The *n-fold Massey product*, $\langle \gamma_1, \ldots, \gamma_n \rangle$, is the set of all homology classes of cycles associated to all possible defining systems.

Since Massey products are preserved by quasi-isomorphisms by [18, 1.5], we can use the simpler ring $\mathcal{E}_a$ to calculate these products for $\mathcal{E}(\mathcal{B}_a)$.

Proposition 4.9. *The homology of $\mathcal{E}_a$ (and hence also of $\mathcal{E}(\mathcal{B}_a)$ and $\mathcal{E}(\mathcal{B}_a)\langle 0 \rangle$) is given by*

1. $H_*\mathcal{E}_a(I_H, I_H) = \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ *with generators* $[\mathrm{id}_H]$ *and* $[m_1^H]$,
2. $H_*\mathcal{E}_a(I_H, I_K) = 0$,
3. $H_*\mathcal{E}_a(I_H, I_\mathbb{T}) = \mathbb{Q}[0]$ *with generator* $[f_0^H]$,
4. $H_*\mathcal{E}_a(I_\mathbb{T}, I_H) = \mathbb{Q}[1]$ *with generator* $[\widetilde{g}_1^H]$, *and*
5. $H_*\mathcal{E}_a(I_\mathbb{T}, I_\mathbb{T}) = (\bigoplus_{n \geq 0} \mathbb{Q}\mathcal{F}[2n+1]) \oplus \mathbb{Q}[0]$ *with generators* $[i_{2n+1}^H]$ *and* $[\mathrm{id}_\mathbb{T}]$.

The non-trivial products, except the obvious ones with $[\mathrm{id}_H]$ *and* $[\mathrm{id}_\mathbb{T}]$*, and the non-trivial Massey products are given by*

1. $[\widetilde{g}_1^H][f_0^H] = [m_1^H]$,
2. $[f_0^H][\widetilde{g}_1^H] = [i_1^H]$, *and*
3. $\langle [f_0^H], \underbrace{[m_1^H], \ldots, [m_1^H]}_{n}, [\widetilde{g}_1^H] \rangle = \{[-i_{2n+1}^H]\}$.

Note that $[\widetilde{g}_1^H]$ *and* $[f_0^H]$ *generate all of the elements in homology via products and Massey products.*

Proof. The products follow from the fact that $\widetilde{g}_1 f_0 = m_1$ and $f_0 \widetilde{g}_1 = i_1$. Next consider the triple product $\langle [f_0^H], [m_1^H], [\widetilde{g}_1^H] \rangle$. The cycle representatives here are unique. Since $f_0^H m_1^H = 0$, $\mathcal{E}_a(I_H, I_\mathbb{T})_2 = 0$ and g_3^H is the unique element such that $dg_3^H = g_2^H = m_1^H \widetilde{g}_1^H$, this determines the only defining system as the following matrix.

$$\begin{pmatrix} f_0^H & 0 & \\ & m_1^H & g_3^H \\ & & \widetilde{g}_1^H \end{pmatrix}$$

The associated cycle is $-f_0^H g_3^H + 0\widetilde{g}_1^H = -i_3^H$, so $\langle [f_0^H], [m_1^H], [\widetilde{g}_1^H] \rangle = \{[-i_3^H]\}$.

The higher order products follow inductively. At each stage because $\mathcal{E}_a(I_H, I_H)_n$ and $\mathcal{E}_a(I_H, I_\mathbb{T})_n$ are trivial for $n > 1$ and $\mathcal{E}_a(I_\mathbb{T}, I_H)_n$ is generated by one element, there are no choices and hence no indeterminacy. Set $a_{1,1} = f_0^H$, $a_{i,i} = m_1^H$ for $2 \leq i \leq n+1$, and $a_{n+2,n+2} = \widetilde{g}_1^H$. Since $f_0^H m_1^H = 0$ and $m_1^H m_1^H = 0$, the only choice for $a_{i,j}$ is 0 for $i < j$ and $j \neq n+2$. Hence the only non-zero entries in the defining system are $a_{i,i}$ for $1 \leq i \leq n+2$ and $a_{i,n+2}$ for $1 < i \leq n+2$. Then the only choice for $a_{i,n+2}$ is $g_{2n-2i+5}^H$ for $i < n+2$. Since $-a_{1,1}a_{2,n+2} = -f_0^H g_{2n+1}^H = -i_{2n+1}^H$, the conclusion follows. □

5. The endomorphism ring for $\mathbb{T}$-spectra

In this section we consider the endomorphism ring $\mathcal{E}(\mathcal{B}_t)$ associated to $\mathbb{T}$-spectra in Corollary 3.9. As with $\mathcal{E}(\mathcal{B}_a)$, we modify $\mathcal{E}(\mathcal{B}_t)$ to define a quasi-isomorphic ring $\mathcal{E}_t$. One property of $\mathcal{E}_t$ is that $\mathcal{E}_t(H, \mathbb{T})$ is concentrated in degree zero. This forces certain relations to hold and thus makes it easier to construct a zig-zag of quasi-isomorphisms between $\mathcal{E}_a$ and $\mathcal{E}_t$. Another property of $\mathcal{E}_t$ is that in degree zero $(\mathcal{E}_t)_0 \cong H_0\mathcal{E}_t$. This gives a place to start the construction of the maps. This is carried out in Section 6.

We first modify $\mathcal{E}(\mathcal{B}_t)$ by taking its (-1)-connected cover $\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle$ as in Definition 4.3. We next modify $\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle$ so that degree zero agrees with its homology. This uses a Postnikov approximation (Definition 5.3); its general properties are stated in Propositions 5.4 and 5.5 and are applied to this specific case in Proposition 5.6. Finally, we define $\mathcal{E}_t$ as a sub-ring of the resulting differential graded ring with many objects. This ring $\mathcal{E}_t$ then has the properties mentioned above.

We first recall the homology of $\mathcal{E}(\mathcal{B}_t)$ from [5, 2.1.6]. Of course, it is isomorphic as a ring to the homology of $\mathcal{E}(\mathcal{B}_a)$ stated in Proposition 4.9 because the triangulated equivalence of $\mathrm{Ho}(\mathbb{T}\text{-spectra})$ and $\mathcal{D}(\mathcal{A})$ from [5, 5.6.1] can be chosen in such a way that the geometric basic cells in $\mathcal{B}_t$ are taken to the algebraic basic cells in $\mathcal{B}_a$. Because of this isomorphism we use the

notation $\widehat{x}$ to denote the element in $H_*\mathcal{E}(\mathcal{B}_t)$ corresponding to $[x]$ in $H_*\mathcal{E}_a$. We also delete most of the superscripts H. We label the elements in $\mathcal{B}_t$ by their corresponding subgroup in $\mathbb{T}$.

Proposition 5.1. *The homology of $\mathcal{E}(\mathcal{B}_t)$ is given by*

1. $H_*\mathcal{E}(\mathcal{B}_t)(H, H) = \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ *with generators* $\widehat{\mathrm{id}_H}$ *and* $\widehat{m_1}$,
2. $H_*\mathcal{E}(\mathcal{B}_t)(H, K) = 0$,
3. $H_*\mathcal{E}(\mathcal{B}_t)(H, \mathbb{T}) = \mathbb{Q}[0]$ *with generator* $\widehat{f_0}$,
4. $H_*\mathcal{E}(\mathcal{B}_t)(\mathbb{T}, H) = \mathbb{Q}[1]$ *with generator* $\widehat{g_1}$ *and*
5. $H_*\mathcal{E}(\mathcal{B}_t)(\mathbb{T}, \mathbb{T}) = (\bigoplus_{n \geq 0} \mathbb{Q}\mathcal{F}[2n + 1]) \oplus \mathbb{Q}[0]$ *with generators* $\widehat{i^H_{2n+1}}$ *and* $\widehat{\mathrm{id}_\mathbb{T}}$.

The non-trivial products, except the obvious ones with $\widehat{\mathrm{id}_H}$ *and* $\widehat{\mathrm{id}_\mathbb{T}}$ *are given by*

1. $\widehat{g_1^H}\,\widehat{f_0^H} = \widehat{m_1^H}$,
2. $\widehat{f_0^H}\,\widehat{g_1^H} = \widehat{i_1^H}$, *and*

The next step is to consider the (-1)-connected cover of $\mathcal{E}(\mathcal{B}_t)$. Since the homology of $\mathcal{E}(\mathcal{B}_t)$ is concentrated in non-negative degrees, $\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle \longrightarrow \mathcal{E}(\mathcal{B}_t)$ is a quasi-isomorphism.

Proposition 5.2. *The map* $\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle \longrightarrow \mathcal{E}(\mathcal{B}_t)$ *is a quasi-isomorphism of differential graded rings with objects $\mathcal{B}_t$. Thus, there is a Quillen equivalence between the respective categories of modules.*

$$\mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$$

The purpose of the next step is to construct a ring $\overline{\mathcal{R}}$ quasi-isomorphic to $\mathcal{E}(\mathcal{B}_t)\langle 0 \rangle$ such that $\overline{\mathcal{R}}_0 \cong H_0(\overline{\mathcal{R}})$. This involves a Postnikov approximation which is obtained by killing homology groups above degree zero. The next definition gives a functorial definition of this approximation which uses the small object argument; see [3, 7.12].

Consider the functor $\mathrm{Ev}_{(H,K)}$ from rational differential graded rings with objects $\mathcal{B}_t$ to rational chain complexes which is evaluation at (H, K). Denote the left adjoint of $\mathrm{Ev}_{(H,K)}$ by $T_{(H,K)}$. $T_{(H,H)}M$ is the tensor algebra on M at (H, H), $\mathbb{Q}[0]$ at (K, K) for $H \neq K$ and trivial everywhere else.

Definition 5.3. Given a rational differential graded ring $\mathcal{R}$ with objects $\mathcal{B}_t$ a Postnikov approximation of $\mathcal{R}$ is given by the factorization $\mathcal{R} \xrightarrow{i} P_0\mathcal{R} \longrightarrow 0$ produced by the small object argument for the set of maps $I = \{T_{(H,K)}(\mathbb{Q}[n] \longrightarrow D^{n+1})\}$ for $n \geq 1$ and all pairs $(H, K) \in \mathcal{B}_t \times \mathcal{B}_t$.

Since the map $i \colon \mathcal{R} \longrightarrow P_0\mathcal{R}$ is constructed by pushouts of the maps in I and $P_0\mathcal{R} \longrightarrow 0$ has the right lifting property with respect to the maps in I, these maps have the following properties.

Proposition 5.4. *The map* $\mathcal{R} \xrightarrow{i} P_0\mathcal{R}$ *is an isomorphism in degrees less than or equal to one and hence induces an isomorphism in homology in non-positive degrees. The map* $P_0\mathcal{R} \longrightarrow 0$ *induces an isomorphism in homology in positive degrees.*

By applying the small object argument one more time we factor the map $\mathcal{R} \xrightarrow{i} P_0\mathcal{R}$. Construct the factorization $\mathcal{R} \xrightarrow{\bar{i}} \overline{\mathcal{R}} \xrightarrow{\bar{p}} P_0\mathcal{R}$ by using the small object argument with respect to the set $J = \{T_{(H,K)}(0 \longrightarrow D^n)\}$ for $n \geq 2$ and all pairs $(H, K) \in \mathcal{B}_t \times \mathcal{B}_t$. Again by construction, since the maps in J are injective quasi-isomorphisms and isomorphisms in non-positive degrees and the map $\mathcal{R} \xrightarrow{i} P_0\mathcal{R}$ has the above listed properties, $\bar{i}$ and $\bar{p}$ have the following properties.

Proposition 5.5. *The map* $\mathcal{R} \xrightarrow{\bar{i}} \overline{\mathcal{R}}$ *is a quasi-isomorphism which is an isomorphism in non-positive degrees.* $\overline{\mathcal{R}} \xrightarrow{\bar{p}} P_0\mathcal{R}$ *is an epimorphism and in non-positive degrees is a quasi-isomorphism and an isomorphism.*

Proof. Since $\bar{p}$ has the lifting property with respect to J, $\mathrm{Ev}_{(H,K)}\,\bar{p}$ is an epimorphism in degrees greater than one. Since $\mathcal{R} \xrightarrow{i} P_0\mathcal{R}$ is an isomorphism in degrees less than two and $i = \bar{p}\bar{i}$, it follows that $\bar{p}$ is also an epimorphism in degrees less than two.   □

We now apply these general constructions to $\mathcal{E}(\mathcal{B}_t)\langle 0\rangle$. For ease of notation, let $\mathcal{E}'_t$ stand for $\mathcal{E}(\mathcal{B}_t)\langle 0\rangle$. Since $\mathcal{E}'_t$ is (-1)-connected, it is trivial in negative degrees and its homology is concentrated in non-negative degrees. Hence, $P_0\mathcal{E}'_t$ is trivial in negative degrees and its homology is concentrated in degree zero. Let $\mathcal{E}'_t \xrightarrow{\bar{i}} \overline{\mathcal{E}'_t} \xrightarrow{\bar{p}} P_0\mathcal{E}'_t$ be the factorization constructed above for $\mathcal{R} = \mathcal{E}'_t$. Consider $H_0\mathcal{E}'_t \cong H_0\mathcal{E}(\mathcal{B}_t)$ as a differential graded ring with objects $\mathcal{B}_t$ which is concentrated in degree zero. From Proposition 5.1, $H_0\mathcal{E}(\mathcal{B}_t)$ is the free $\mathbb{Q}$-algebra with objects $\mathcal{B}_t$ generated by a one dimensional vector space of morphisms from H to $\mathbb{T}$ for each $H \in \mathcal{F}$. Since $H_0\mathcal{E}(\mathcal{B}_t) \cong H_0\mathcal{E}'_t \cong H_0 P_0\mathcal{E}'_t$, one can construct a quasi-isomorphism of differential graded rings $q\colon H_0\mathcal{E}'_t \longrightarrow P_0\mathcal{E}'_t$. Define $\mathcal{E}''_t$ as the differential graded ring with objects $\mathcal{B}_t$ which is the pullback of $\bar{p}$ and q.

$$
\begin{array}{ccc}
\mathcal{E}''_t & \xrightarrow{q'} & \overline{\mathcal{E}'_t} \\
\downarrow & & \downarrow{\scriptstyle \bar{p}} \\
H_0\mathcal{E}'_t & \xrightarrow{q} & P_0\mathcal{E}'_t
\end{array}
$$

Proposition 5.6. *The map $q'\colon \mathcal{E}''_t \longrightarrow \overline{\mathcal{E}'_t}$ is a quasi-isomorphism. Moreover, $(\mathcal{E}''_t)_0 \cong H_0\mathcal{E}''_t \cong H_0\mathcal{E}'_t$ and $(\mathcal{E}''_t)_n = 0$ for $n < 0$.*

Proof. $\mathcal{E}''_t(H,K)$ is the pullback of differential graded modules of the maps $\mathrm{Ev}_{(H,K)}\,\bar{p}$ and $\mathrm{Ev}_{(H,K)}\,q$. Each map $\mathrm{Ev}_{(H,K)}\,\bar{p}$ is an epimorphism by Proposition 5.5 and each map $\mathrm{Ev}_{(H,K)}\,q$ is a quasi-isomorphism. The model category of differential graded modules is right proper since every object is fibrant. So $\mathcal{E}''_t(H,K) \longrightarrow \overline{\mathcal{E}'_t}(H,K)$ is a quasi-isomorphism because it is the pullback of a weak equivalence across a fibration. Since $\bar{p}$ is an isomorphism in non-positive degrees by Proposition 5.5, $(\mathcal{E}''_t)_n$ is isomorphic to $(H_0\mathcal{E}'_t)_n$ for $n \leq 0$. Note that since q' is a quasi-isomorphism this also implies that $d_1\colon (\mathcal{E}''_t)_1 \longrightarrow (\mathcal{E}''_t)_0$ is trivial.   □

We make one more modification to $\mathcal{E}''_t$. Define $\mathcal{E}_t$ as the sub-differential graded ring with objects $\mathcal{B}_t$ which differs from $\mathcal{E}''_t$ only by setting $\mathcal{E}_t(H,\mathbb{T}) = \mathcal{E}''_t(H,\mathbb{T})_0$ concentrated in degree zero for each $H \in \mathcal{F}$ and $\mathcal{E}_t(H,K) = 0$ for H, K distinct finite subgroups. This defines a differential graded ring since $\mathcal{E}''_t(H,\mathbb{T})_0$ is a retract of $\mathcal{E}''_t(H,\mathbb{T})$ because d_1 is trivial. It follows that the map $\mathcal{E}_t \longrightarrow \mathcal{E}''_t$ is a quasi-isomorphism and $\mathcal{E}_t$ has the properties listed in Proposition 5.6 for $\mathcal{E}''_t$. Combining this with Propositions 5.2 and 5.5 gives the following.

Proposition 5.7. *The composite map $\mathcal{E}_t \longrightarrow \mathcal{E}''_t \longrightarrow \overline{\mathcal{E}'_t}$ is a quasi-isomorphism of differential graded rings with objects $\mathcal{B}_t$. Combining this with the quasi-isomorphisms $\mathcal{E}(\mathcal{B}_t)\langle 0\rangle \cong \mathcal{E}'_t \longrightarrow \overline{\mathcal{E}'_t}$ and $\mathcal{E}(\mathcal{B}_t)\langle 0\rangle \longrightarrow \mathcal{E}(\mathcal{B}_t)$ gives a zig-zag of quasi-isomorphisms between $\mathcal{E}_t$ and $\mathcal{E}(\mathcal{B}_t)$. Thus, there is a zig-zag of Quillen equivalences between the respective categories of modules.*

$$\mathcal{M}od\text{-}\mathcal{E}_t \simeq_Q \mathcal{M}od\text{-}\mathcal{E}(\mathcal{B}_t)$$

Moreover, $(\mathcal{E}_t)_0 \cong H_0(\mathcal{E}_t)$, $(\mathcal{E}_t)_n = 0$ for $n < 0$, $\mathcal{E}_t(H,K) = 0$ and $\mathcal{E}_t(H,\mathbb{T})$ is concentrated in degree zero.

6. The quasi-isomorphism between $\mathcal{E}_a$ and $\mathcal{E}_t$

In this section we construct a differential graded ring $\mathcal{S}$ with objects $\mathcal{B}_a$ and a zig-zag of quasi-isomorphisms $\varphi\colon \mathcal{E}_a \xleftarrow{\eta} \mathcal{S} \longrightarrow \mathcal{E}_t$. Note that there is an obvious bijection between the

object sets $\mathcal{B}_a$ and $\mathcal{B}_t$ which is the first condition required for such a quasi-isomorphism of rings with many objects. In this section we label the elements of both these sets by the corresponding subgroup of $\mathbb{T}$. Recall that $\mathcal{E}_a$ is generated as an algebra by the elements f_0^H and g_{2n+1}^H with the relations forced by $\mathcal{E}_a(H, K)$ being trivial, $\mathcal{E}_a(H, \mathbb{T})$ being trivial above degree zero and $\mathcal{E}_a(H, H)$ being trivial above degree one. Since $\mathcal{E}_t(H, H)$ does not necessarily share this property, we must use the auxiliary ring $\mathcal{S}$ which is basically constructed from $\mathcal{E}_a$ by removing the relations forced by $\mathcal{E}_a(H, H)$ being trivial above degree one.

Proposition 6.1. *There is a differential graded ring $\mathcal{S}$ with objects $\mathcal{B}_a$ and quasi-isomorphisms $\mathcal{E}_a \xleftarrow{\eta} \mathcal{S} \xrightarrow{\varphi} \mathcal{E}_t$ of differential graded rings which induce the obvious bijections $\mathcal{B}_a \xleftarrow{\mathrm{id}} \mathcal{B}_a \longrightarrow \mathcal{B}_t$.*

Combining Propositions 4.7, 5.7 and 6.1 and Corollaries 3.7 and 3.9 gives our main result.

Corollary 6.2. *There is a zig-zag of Quillen equivalences between rational $\mathbb{T}$-equivariant spectra and differential graded objects in $\mathcal{A}$.*

$$\mathbb{T}\text{-}spectra \simeq_Q dg\mathcal{A}$$

We first define the intermediary ring $\mathcal{S}$ and the map $\eta \colon \mathcal{S} \longrightarrow \mathcal{E}_a$. As mentioned above, $\mathcal{S}(H, H)$ is not required to be trivial above degree one; so we remove the relation $g_{2k+1}^H f_0^H = 0$. Then we must add generators $s_{i,j}^H$ to ensure that these products are still trivial in homology.

Definition 6.3. Define the differential graded ring $\mathcal{S}$ with objects $\mathcal{B}_a$ as the ring generated by elements
1. $f_0^H \in \mathcal{S}(H, \mathbb{T})_0$ with $d(f_0^H) = 0$,
2. $g_{2k+1}^H \in \mathcal{S}(\mathbb{T}, H)_{2k+1}$ for $k \geq 0$ with $d(g_1^H) = 0$ and $d(g_{2k+1}^H) = g_1^H f_0^H g_{2k-1}^H$ for $k > 0$ and
3. $s_{2m+1,2n}^H \in \mathcal{S}(H, H)_{2n}$ for $m \geq 1$ and $n \geq m + 1$ with $d(s_{2m+1,2m+2}^H) = g_{2m+1}^H f_0^H$ and $d(s_{2m+1,2n}^H) = s_{2m+1,2n-2}^H g_1^H f_0^H$ for $n > m + 1$

with relations generated by $x^H y^K = 0$ for any elements with $H \neq K$, $f_0^H s_{2m+1,2n}^H = 0$ and $f_0^H g_{2k+1}^H f_0^H = 0$. The relations are equivalent to setting $\mathcal{S}(H, \mathbb{T}) \cong \mathbb{Q}[0]$ and $\mathcal{S}(H, K) = 0$.

As mentioned in Definition 4.6, for ease of notation in this section we denote $\widetilde{g}_1$ in $\mathcal{E}_a(\mathbb{T}, H)$ by g_1. The element g_1 in $\mathcal{E}(\mathcal{B}_a)$ does not appear anywhere in this section. Define $\eta \colon \mathcal{S} \longrightarrow \mathcal{E}_a$ as the ring homomorphism determined by $\eta(f_0^H) = f_0^H$, $\eta(g_{2k+1}^H) = g_{2k+1}^H$ and $\eta(s_{2m+1,2n}^H) = 0$.

Proposition 6.4. *The map $\eta \colon \mathcal{S} \longrightarrow \mathcal{E}_a$ is a quasi-isomorphism of differential graded rings.*

Proof. For ease of notation we delete the superscripts H and implicitly apply these steps for each finite subgroup H. Because there is no interaction between different finite subgroups, this should not cause any confusion. Since η takes the relations in $\mathcal{S}$ to zero in $\mathcal{E}_a$, it is a ring homomorphism. It also commutes with the differential since in $\mathcal{E}_a$ we have that $d(g_{2k+1}) = g_{2k} = g_1 f_0 g_{2k-1}$ and $g_{2k+1} f_0 = 0$ for $k > 0$. The maps $\mathcal{S}(H, K) \xrightarrow{\eta} \mathcal{E}_a(H, K)$, $\mathcal{S}(H, \mathbb{T}) \xrightarrow{\eta} \mathcal{E}_a(H, \mathbb{T})$ and $\mathcal{S}(\mathbb{T}, \mathbb{T}) \xrightarrow{\eta} \mathcal{E}_a(\mathbb{T}, \mathbb{T})$ are isomorphisms.

We next show that $H_* \mathcal{S}(H, H) \cong \mathbb{Q}[0] \oplus \mathbb{Q}[1]$, generated by $[\mathrm{id}_H]$ and $[g_1 f_0]$. Below is a table of generating elements in degree zero through seven in $\mathcal{S}(H, H)$ and $\mathcal{S}(\mathbb{T}, H)$.

	0	1	2	3	4	5	6	7
$\mathcal{S}(H, H)$	id_H	$g_1 f_0$		$g_3 f_0$	$s_{3,4}$	$s_{3,4} g_1 f_0$ $g_5 f_0$	$s_{3,6}$ $s_{5,6}$	$s_{3,4} g_3 f_0$, $g_7 f_0$ $s_{5,6} g_1 f_0$, $s_{3,6} g_1 f_0$
$\mathcal{S}(\mathbb{T}, H)$		g_1	$g_1 f_0 g_1$	g_3	$g_1 f_0 g_3$ $g_3 f_0 g_1$	g_5 $s_{3,4} g_1$	$g_1 f_0 g_5$, $g_3 f_0 g_3$ $g_5 f_0 g_1$, $s_{3,4} g_1 f_0 g_1$	g_7, $s_{3,4} g_3$ $s_{5,6} g_1$, $s_{3,6} g_1$

These elements have been arranged so that the differential is either zero or takes an element to the element in the corresponding spot one degree below. Here, $d_n \colon \mathcal{S}_n \longrightarrow \mathcal{S}_{n-1}$ is non-zero for

$n = 4, 6$ on $\mathcal{S}(H, H)$ and for $n = 3, 5, 7$ on $\mathcal{S}(\mathbb{T}, H)$. In general, because of the relations in $\mathcal{S}$ any element in $\mathcal{S}(H, H)$ can be written as a sum of elements of the form w or $wg_{2k+1}f_0$ where w is a word in the elements of the set $\{s_{2m+1,2n}\}$ (including the empty word $w = \mathrm{id}_H$.) Again because of the relations, dw has only one term:

$$d(ws_{2m+1,2m+2}) = wg_{2m+1}f_0 \text{ and } d(ws_{2m+1,2n}) = ws_{2m+1,2n-2}g_1f_0 \text{ for } n > m + 1.$$

Hence if w is not the empty word, then w is not a cycle. The elements $wg_{2k+1}f_0$ are cycles; if $k > 0$ then it is also a boundary (of $ws_{2k+1,2k+2}$) or if $k = 0$ and w is not id_H then again it is a boundary since $w = w's_{2m+1,2n}$ and $d(w's_{2m+1,2n+2}) = w's_{2m+1,2n}g_1f_0 = wg_1f_0$. This leaves only id_H and g_1f_0 as generators of homology. Since $m_1 = g_1f_0$ in $\mathcal{E}_a$, we see that $\mathcal{S}(H, H) \xrightarrow{\eta} \mathcal{E}_a(H, H)$ induces an isomorphism in homology.

Finally we show that $H_*\mathcal{S}(\mathbb{T}, H) \cong \mathbb{Q}[1]$ generated by g_1 by showing that $\mathcal{S}(\mathbb{T}, H) \cong (\bigoplus_I D^{n_i}) \oplus \mathbb{Q}[1]$ as a rational chain complex. Every element of $\mathcal{S}(\mathbb{T}, H)$ can be written as a sum of elements of the form wg_i or $wg_jf_0g_k$ where i, j and k are odd numbers and w is a (possibly empty) word in the $s_{2m+1,2n}$ as above. Except for g_1 these module generators fall into certain families $\mathcal{F}_{w,i,j-i}$ indexed by a word, w, and two odd integers, i and $j - i$. The families with $i = 1$ are exceptional, $\mathcal{F}_{w,1,j-1}$ is the set $\{g_{j+1}, g_1f_0g_{j-1}\}$ which generates a complex isomorphic to D^{j+1}. If $i > 1$ and $j - i = 2l + 1$, then $\mathcal{F}_{w,i,j-i}$ generates a complex isomorphic to $l + 1$ copies of $D^{|w|+j+1}$. This set is defined by

$$\mathcal{F}_{w,i,j-i} = \{ws_{i,k}g_{j-k+1} \text{ for } k \text{ even and } i + 1 \leq k \leq j,$$

$$ws_{i,k}g_1f_0g_{j-k-1} \text{ for } k \text{ even and } i + 1 \leq k \leq j - 2, \text{ and } wg_if_0g_{j-i}\}.$$

Since every module generator except g_1 appears in one and only one of these families, this gives the splitting as claimed. Since $\eta(g_1) = g_1$, it follows that $\eta\colon \mathcal{S}(\mathbb{T}, H) \longrightarrow \mathcal{E}_a(\mathbb{T}, H)$ is a quasi-isomorphism. □

To finish the proof of Proposition 6.1 we must construct $\varphi\colon \mathcal{S} \longrightarrow \mathcal{E}_t$. This construction uses Toda brackets in $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$ which we calculate using Massey products. For $\mathcal{R}$ a differential graded ring with objects $\mathcal{G} = \{G_i\}_{i \in I}$, a cycle $a \in \mathcal{R}(G_j, G_i)_p$ induces a map of right $\mathcal{R}$-modules, $a\colon \Sigma^p F^{\mathcal{R}}_{G_j} \longrightarrow F^{\mathcal{R}}_{G_i}$, by left multiplication (or post-composition) where $F^{\mathcal{R}}_{G_i}(-) = \mathcal{R}(-, G_i)$ is the free right $\mathcal{R}$-module represented by G_i. The associated homotopy class of maps $[a]$ is determined by the homology class represented by a in the homology of $\mathcal{R}$. This translates between Massey products, which are homology classes in $\mathcal{R}$, and Toda brackets, which are homotopy classes of right $\mathcal{R}$-module maps. Notice, if $a \in \mathcal{R}(G_j, G_i)_p$ and $b \in \mathcal{R}(G_i, G_k)_q$ are cycles then extra suspensions are required to compose these elements; the associated composition is $b(\Sigma^q a)\colon \Sigma^{p+q}G_j \longrightarrow \Sigma^q G_i \longrightarrow G_k$.

Starting with the Massey products for $\mathcal{E}_a$ we construct elements of the Toda brackets for $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_a)$ in Proposition 6.5. See Definition A.2 for the definition of higher Toda brackets. We denote the Toda brackets by $\langle - \rangle_T$, to differentiate them from the Massey products. Since Toda brackets are invariant under exact equivalences by Theorem A.3, these Toda brackets agree with those for $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$. This follows because Quillen equivalences induce triangulated equivalences [20, I.4.3], $\mathcal{M}od\text{-}\mathcal{E}_t$ is Quillen equivalent to $\mathbb{T}$-spectra by Proposition 5.7 and Corollary 3.9, $\mathcal{M}od\text{-}\mathcal{E}_a$ is Quillen equivalent to $dg\mathcal{A}$ by Proposition 4.7 and Corollary 3.7 and $\mathrm{Ho}(\mathbb{T}\text{-spectra})$ is triangulated equivalent to $\mathcal{D}(\mathcal{A})$ by [5, 5.6.1]. The Toda brackets in $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$ then give us the structure we need to construct φ.

Proposition 6.5. *The following Toda brackets in* $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_a)$ *agree up to sign with the Massey products in* $H_*\mathcal{E}_a$ *from Proposition 4.9.*

$$\langle [f_0^H], [m_1^H], \Sigma[m_1^H], \ldots, \Sigma^{n-1}[m_1^H], \Sigma^n[g_1^H] \rangle_T = \{[f_0^H g_{2n+1}^H]\} = \{[i_{2n+1}^H]\}$$

The sign difference here corresponds to the different sign conventions used for Massey products and Toda brackets.

Proof. For ease of notation we fix a finite subgroup H and delete the superscripts H. We first consider the three-fold Toda bracket. Starting with the defining system for the Massey product $\langle [f_0], [m_1], [g_1] \rangle$ we construct an element of the Toda bracket $\langle [f_0], [m_1], \Sigma[g_1] \rangle_T$. By Definition A.2 such a Toda bracket is the set of homotopy classes of compositions $\Sigma^3 F_{\mathbb{T}}^{\mathcal{E}_a} \xrightarrow{\beta} X \xrightarrow{\alpha} F_{\mathbb{T}}^{\mathcal{E}_a}$ where the homotopy type of X is a 2-filtered object in $\{[m_1]\}$; see Definition A.1. The cofiber of m_1 is such an object. Explicitly, set $X = \Sigma^2 F_H^{\mathcal{E}_a} \ltimes_{m_1} F_H^{\mathcal{E}_a}$ with underlying graded object $\Sigma^2 F_H^{\mathcal{E}_a} \oplus F_H^{\mathcal{E}_a}$ and with differential given by $d(x, y) = (dx, dy - m_1 x)$. Then the defining system found in Proposition 4.9 determines the other two maps as well, $\alpha(x, y) = f_0 y$ and $\beta(z) = (g_1 z, g_3 z)$. Here we are using the sign convention that $d_{\Sigma C} = -d_C$. One can check that $[\sigma_X \beta] = [g_1]$ and $[\alpha \sigma_X'] = [f_0]$. Due to the sparseness of elements in $\mathcal{E}_a$, one can check that α, β are the unique maps satisfying the definition of the Toda bracket. Since $\Sigma^3 F_{\mathbb{T}}^{\mathcal{E}_a}, X$ and $F_{\mathbb{T}}^{\mathcal{E}_a}$ are cofibrant and fibrant $\mathcal{E}_a$-modules these are also the unique such homotopy classes. Hence the Toda bracket is the one element set $\{[f_0 g_3] = [i_3]\}$.

As with the three-fold Massey product, the higher Massey products also agree up to sign with the higher Toda brackets. First, define the iterated cofiber

$$X_n = \Sigma^{2n} F_H^{\mathcal{E}_a} \ltimes_{m_1} \Sigma^{2n-2} F_H^{\mathcal{E}_a} \ltimes_{m_1} \ldots \ltimes_{m_1} F_H^{\mathcal{E}_a}$$

with differential $d(x_1, x_2, \ldots, x_{n+1}) = (dx_1, dx_2 - m_1 x_1, \ldots, dx_{n+1} - m_1 x_n)$. Note here $d^2 = 0$ because $m_1^2 = 0$. By Lemma A.4, X_n is an element of $\{[m_1], \Sigma[m_1], \ldots, \Sigma^{n-1}[m_1]\}$. Then the associated defining system determines the maps $\Sigma^{2n+1} F_{\mathbb{T}}^{\mathcal{E}_a} \xrightarrow{\beta} X_n \xrightarrow{\alpha} F_{\mathbb{T}}^{\mathcal{E}_a}$ as $\beta(z) = (g_1 z, g_3 z, \ldots, g_{2n+1} z)$ and $\alpha(x_1, \ldots, x_{n+1}) = f_0 x_{n+1}$. This shows that the composite, $[f_0 g_{2n+1}] = [i_{2n+1}]$ is in the Toda bracket $\langle [f_0], [m_1], \Sigma[m_1], \ldots, \Sigma^{n-1}[m_1], \Sigma^n[g_1] \rangle_T$. One can check that any other defining system would produce a Toda bracket and vice versa, but again the sparseness of elements in $\mathcal{E}_a$ forces this to be the only element in the Toda bracket as well as the only defining system. $\qquad\square$

Proof of Proposition 6.1. We are left with constructing $\varphi \colon \mathcal{S} \longrightarrow \mathcal{E}_t$. We proceed by induction on the degree. We exploit the fact that Toda brackets in $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$ and $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_a)$ agree to construct φ. At each stage certain products must be non-trivial because of the non-trivial Toda brackets. This then provides non-trivial targets for φ of the elements g_{2k+1}^H in $\mathcal{S}$. The products $g_{2k+1}^H f_0^H$ are trivial in homology (for $k > 0$) so they must be boundaries. This produces targets for φ of the elements $s_{2k+1,2l}^H$ in $\mathcal{S}$. We can then extend φ to a ring homomorphism because the images of the relations in $\mathcal{S}$ between these generators hold in $\mathcal{E}_t$ as well since $\mathcal{E}_t(H, \mathbb{T})$ is concentrated in degree zero. By construction φ is then a map of differential graded objects and we show inductively that it is also a quasi-isomorphism.

Again, we mostly delete the superscripts H and implicitly apply these steps for each finite subgroup. Since $H_* \mathcal{E}_t$ is isomorphic to $H_* \mathcal{E}(\mathcal{B}_t)$, we use the notation for elements in $H_* \mathcal{E}_t$ from Proposition 5.1. To further ease notation, if y is the boundary of x, i.e. $dx = y$, then we say that x is a *primitive* of y.

The images of id_H and $\mathrm{id}_{\mathbb{T}}$ in $\mathcal{S}$ are forced to be the respective identity elements in $\mathcal{E}_t$. We next determine the images of f_0 and g_1. Let f_0' be an element in $\mathcal{E}_t(H, \mathbb{T})_0$ such that $[f_0'] = \widehat{f_0}$ in $H_0 \mathcal{E}_t(H, \mathbb{T})$. Let g_1' be a cycle such that $[g_1'] = \widehat{g_1}$ in $H_1 \mathcal{E}_t(\mathbb{T}, H)$. Then set $\varphi(f_0) = f_0'$ and $\varphi(g_1) = g_1'$. Since $\mathcal{E}_t(H, \mathbb{T})$ is concentrated in degree zero by Proposition 5.7, $f_0' g_1' f_0'$ is trivial. Thus the relations between f_0 and g_1 in $\mathcal{S}$ also hold in $\mathcal{E}_t$ so we may extend φ to products of g_1 and f_0 using the product structure in $\mathcal{E}_t$. Since $[g_1'][f_0'] = \widehat{m_1}$ and $[f_0'][g_1'] = \widehat{i_1}$ are non-trivial

in $H_*\mathcal{E}_t$ by Propostions 5.1 and 5.7, the images $\varphi(g_1 f_0)$ and $\varphi(f_0 g_1)$ must be non-trivial cycles. Hence φ induces a quasi-isomorphism in degrees zero and one.

Before turning to the general induction step, we consider degrees two and three. It may be useful to refer to the tabulation of elements of $\mathcal{S}(H, H)$ and $\mathcal{S}(\mathbb{T}, H)$ given in the proof of Proposition 6.4. There we also showed the other components of $\mathcal{S}$ are isomorphic to $\mathcal{E}_a$, see Definition 4.6; as vector spaces $\mathcal{S}(H, K) = 0$, $\mathcal{S}(H, \mathbb{T})$ is generated by f_0 and $\mathcal{S}(\mathbb{T}, \mathbb{T})$ is generated by the elements $\mathrm{id}_\mathbb{T}, f_0 g_1, f_0 g_3, \ldots$. As mentioned above, Toda brackets agree for $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_a)$ and $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$ by Theorem A.3. Since $\langle [f_0], [m_1], \Sigma[g_1] \rangle_T = \{[i_{2n+1}]\}$ by Proposition 6.5 in $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_a)$, there must be maps $\Sigma^3 F_\mathbb{T}^{\mathcal{E}_t} \xrightarrow{\beta'} X' \xrightarrow{\alpha'} F_\mathbb{T}^{\mathcal{E}_t}$ such that $[\alpha'\beta'] = \widehat{i_3}$ is in $\langle \widehat{f_0}, \widehat{m_1}, \Sigma \widehat{g_1} \rangle_T$ in $\mathrm{Ho}(\mathcal{M}od\text{-}\mathcal{E}_t)$. Here X' must be a 2-filtered object in $\{\widehat{m_1}\}$, so define X' as the cofiber of $g_1' f_0'$, similar to X in the proof of Proposition 6.5 above. Then since $\mathcal{E}_t(H, \mathbb{T})_n$ is trivial for $n > 0$ the map α' must be $\alpha'(x, y) = f_0' y$. If $g_1' f_0' g_1' = 0$ then one could set $\beta'(z) = (g_1' z, 0)$. Then the composite $[\alpha'\beta']$ would be trivial. Since this Toda bracket does not contain the trivial map, this means $g_1' f_0' g_1'$ must be non-trivial. Since $H_2 \mathcal{E}_t(\mathbb{T}, H) = 0$ it must also be a boundary, let g_3' be one of its primitives in $\mathcal{E}_t(\mathbb{T}, H)_3$. Then set $\beta'(z) = (g_1' z, g_3' z)$. The maps α' and β' satisfy the definition of the Toda bracket, so by Proposition 6.5 we must have $[\alpha'\beta'] = [f_0' g_3'] = \widehat{i_3}$. Set $\varphi(g_3) = g_3'$.

We next inductively determine the images of $s_{3,2n}$ with $n \geq 2$. Since $f_0' g_1' f_0'$ is in the trivial group $\mathcal{E}_t(H, \mathbb{T})_1$, $d(g_3' f_0') = (dg_3') f_0' = g_1' f_0' g_1' f_0' = 0$ and the element $g_3' f_0'$ is a cycle. It is also a boundary since $H_3 \mathcal{E}_t(H, H) = 0$ by Proposition 5.1 and 5.7; let $s_{3,4}'$ be one of its primitives and set $\varphi(s_{3,4}) = s_{3,4}'$. Assume that elements $s_{3,2k}'$ have been chosen which are primitives of $s_{3,2k-2}' g_1' f_0'$ for $2 < k < n$. Again, since $f_0' g_1' f_0'$ is trivial and $d(s_{3,2n-2}' g_1' f_0') = s_{3,2n-4}' g_1' f_0' g_1' f_0'$ the element $s_{3,2n-2}' g_1' f_0'$ is a cycle. Since $H_{2n-1} \mathcal{E}_t(H, H) = 0$ for $n > 1$ by Proposition 5.1 and 5.7, it is also a boundary. Let $s_{3,2n}'$ be one of its primitives and set $\varphi(s_{3,2n}) = s_{3,2n}'$.

We can now extend φ to the subring of $\mathcal{S}$ generated by f_0, g_1, g_3 and $s_{3,2n}$ for $n \geq 2$. This is possible because the relations among these elements in $\mathcal{S}$ are also satisfied in $\mathcal{E}_t$. Specifically, since $\mathcal{E}_t(H, \mathbb{T})$ is concentrated in degree zero $f_0' s_{3,2n}' = 0$, $f_0' g_1' f_0' = 0$ and $f_0' g_3' f_0' = 0$. By construction φ also commutes with the differential on this subring. The homology in degrees two and three of $\mathcal{E}_t$ is generated by the elements $[f_0^{H'} g_3^{H'}] = \widehat{i_3^H}$ in $H_3 \mathcal{E}_t = H_3 \mathcal{E}_t(\mathbb{T}, \mathbb{T})$. In Proposition 6.4 we showed $\eta \colon \mathcal{S} \longrightarrow \mathcal{E}_a$ is a quasi-isomorphism, so Propostion 4.6 shows that the elements $f_0^H g_3^H$ generate the homology in degrees two and three for $\mathcal{S}$ as well. Thus, φ induces a quasi-isomorphism through degree 3.

Assume by induction that $\varphi(g_{2k+1}) = g_{2k+1}'$ and $\varphi(s_{2k+1,2l}) = s_{2k+1,2l}'$ for $k \leq n$ and $l \geq k+1$, and that the extension of φ to the subring generated by these elements induces a quasi-isomorphism through degree $2n+1$. Since $g_1' f_0' g_1' f_0' = 0$, an $\mathcal{E}_t$-module

$$X_n' \in \{\widehat{m_1}, \Sigma \widehat{m_1}, \ldots, \Sigma^{n-1} \widehat{m_1}\}$$

exists, and can be built similarly to X_n in Proposition 6.5. Since $\mathcal{E}_t(H, \mathbb{T})_n = 0$ for $n > 0$, the map $\alpha' \colon X_n' \longrightarrow F_\mathbb{T}^{\mathcal{E}_t}$ is determined as $\alpha'(x_1, \ldots, x_{n+1}) = f_0' x_{n+1}$. As with the three-fold Toda bracket if $g_1' f_0' g_{2n+1}'$ were trivial then we could set $\beta'(z) = (g_1' z, g_3' z, \ldots, g_{2n+1}' z, 0)$. But then the composite $\alpha'\beta'$ would be trivial, which is not allowed because the $n+2$-fold Toda bracket from Proposition 6.5 does not contain the trivial map. Hence $g_1' f_0' g_{2n+1}'$ is non-trivial. Since $H_{2n+2} \mathcal{E}_t(\mathbb{T}, H) = 0$, it must also be a boundary; let $g_{2n+3}' \in \mathcal{E}_t(\mathbb{T}, H)_{2n+3}$ be one of its primitives and set $\beta'(z) = (g_1' z, g_3' z, \ldots, g_{2n+3}' z)$. Then α' and β' satisfy the definition of the Toda Bracket so

$$[\alpha'\beta'] = [f_0' g_{2n+3}'] \in \langle \widehat{f_0}, \widehat{m_1}, \Sigma \widehat{m_1}, \ldots, \Sigma^{n-1} \widehat{m_1}, \Sigma^n \widehat{g_1} \rangle_T = \{\widehat{i_{2n+3}}\}.$$

Set $\varphi(g_{2n+3}) = g_{2n+3}'$.

Arguments similar to those for $s_{3,2k}$ for $k \geq 2$ also apply to the elements $s_{2n+3,2k}$ for $k \geq n+2$, $n > 0$. The element $g'_{2n+3}f'_0$ is a cycle and a boundary since $H_{2n+3}\mathcal{E}_t(H,H) = 0$; let $s'_{2n+3,2n+4}$ be one of its primitives and set $\varphi(s_{2n+3,2n+4}) = s'_{2n+3,2n+4}$. Assume that elements $s'_{2n+3,2k}$ have been chosen which are primitives of $s'_{2n+3,2k-2}g'_1f'_0$ for $n+2 < k < l$. Since $f'_0g'_1f'_0$ is trivial and $H_{2n-1}\mathcal{E}_t(H,H) = 0$, the element $s'_{2n+3,2l-2}g'_1f'_0$ is a cycle and a boundary. Let $s'_{2n+3,2l}$ be one of its primitives and set $\varphi(s_{2n+3,2l}) = s'_{2n+3,2l}$.

We can now extend φ to the subring of $\mathcal{S}$ generated by f_0, g_{2k+1} and $s_{2k+1,2l}$ for $k \leq n+1$ and $l \geq k+1$; again, this is possible because $\mathcal{E}_t(H,\mathbb{T})$ is concentrated in degree zero, so the relations hold. Also, by construction φ commutes with the differential on this subring. Since the elements $[f_0^{H'}g_{2n+3}^{H\,'}] = \widehat{i_{2n+3}^H}$ generate $H_{2n+3}\mathcal{E}_t = H_{2n+3}\mathcal{E}_t(\mathbb{T},\mathbb{T})$ and the elements $f_0^H g_{2n+3}^H$ generate $H_{2n+3}\mathcal{S} \cong H_{2n+3}\mathcal{E}_a \cong H_{2n+3}\mathcal{E}_a(\mathbb{T},\mathbb{T})$, we see that φ induces a quasi-isomorphism through degree $2n+3$. Hence by induction we have constructed the quasi-isomorphism $\varphi\colon \mathcal{S} \longrightarrow \mathcal{E}_t$. $\square$

APPENDIX A. HIGHER TODA BRACKETS AND MASSEY PRODUCTS

In this section we show that an exact equivalence of triangulated categories preserves higher Toda brackets. We first make slight modifications to the definitions of higher Toda brackets from [1, 2] so that they apply to an arbitrary triangulated category $\mathcal{T}$. We have kept the same order for the maps in the Toda bracket as in [1], but we have reversed the numbering to match the Massey products numbering.

Definition A.1. An *n-filtered object* X is an object X in $\mathcal{T}$ with maps $* \cong F_0X \xrightarrow{i_0} F_1X \xrightarrow{i_1} \cdots \longrightarrow F_{n-1}X \xrightarrow{i_{n-1}} F_nX \cong X$. Let $F_{k+1}X/F_kX$ denote the cofiber of i_k so that there are triangles, $F_kX \xrightarrow{i_k} F_{k+1}X \xrightarrow{\pi_{k+1}} F_{k+1}X/F_kX \xrightarrow{\delta_k} \Sigma F_kX$. Given a composable sequence of maps, $A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} \cdots A_1 \xrightarrow{f_1} A_0$, an *n*-filtered object X *is an element of* $\{f_1, \ldots, f_{n-1}\}$ if and only if $\Sigma^k A_k \cong F_{k+1}X/F_kX$ and via these isomorphisms $\Sigma^k f_k$ is isomorphic to the composite $F_{k+1}X/F_kX \xrightarrow{\delta_k} \Sigma F_kX \xrightarrow{\Sigma \pi_k} \Sigma(F_kX/F_{k-1}X)$. For $X \in \{f_1, \ldots, f_{n-1}\}$, define σ'_X as the composite $A_0 \cong F_1X \longrightarrow X$ and define σ_X as the composite $X \cong F_nX \xrightarrow{\pi_n} F_nX/F_{n-1}X \cong \Sigma^{n-1}A_{n-1}$.

Definition A.2. Define the *n*-fold Toda bracket $\langle f_1, f_2, \ldots, f_n \rangle_T$ for a sequence of maps

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

as the set of all maps $\theta \in \mathcal{T}(\Sigma^{n-2}A_n, A_0)$ such that there is some $(n-1)$-filtered object $X \in \{f_2, \ldots, f_{n-1}\}$ and maps $\beta\colon \Sigma^{n-2}A_n \longrightarrow X$ and $\alpha\colon X \longrightarrow A_0$ with $\Sigma^{n-2}f_n \cong \sigma_X\beta$, $f_1 \cong \alpha\sigma'_X$ and $\theta \cong \alpha\beta$.

With these definitions it is easy to see that Toda brackets are preserved by exact equivalences of triangulated categories.

Theorem A.3. *Given an exact equivalence of triangulated categories* $\psi\colon \mathcal{T} \longrightarrow \mathcal{T}'$, *then* $\theta \in \langle f_1, \ldots, f_n \rangle_T$ *if and only if* $\psi(\theta) \in \langle \psi(f_1), \ldots, \psi(f_n) \rangle_T$.

Proof. Assume given $\theta \in \langle f_1, \ldots, f_n \rangle_T$. Then $\theta = \alpha\beta$ with $\beta \colon \Sigma^{n-2}A_n \longrightarrow X$ and $\alpha \colon X \longrightarrow A_0$ with $X \in \{f_2, \ldots, f_{n-1}\}$. Since ψ is exact, $\psi(X)$, with filtrations $\psi(F_k X)$, is an element of $\{\psi(f_2), \ldots, \psi(f_{n-1})\}$. Hence $\psi(\theta) = \psi(\alpha)\psi(\beta)$ is an element of $\langle \psi(f_1), \ldots, \psi(f_n) \rangle_T$. For the other direction, apply the same arguments to the inverse equivalence ψ^{-1}. $\square$

The following manipulations are useful for constructing filtered objects. Let C_α denote the cofiber of the map α.

Lemma A.4. *Assume given a composable sequence of maps $f_i \colon A_i \longrightarrow A_{i-1}$.*

1. *If $X \in \{f_2, \ldots, f_k\}$ and $\alpha \colon X \longrightarrow A_0$, then $C_\alpha \in \{\alpha\sigma'_X, f_2, \ldots, f_k\}$.*
2. *If $Y \in \{f_1, \ldots, f_{k-1}\}$ and $\alpha \colon \Sigma^{k-1}A_k \longrightarrow Y$, then $C_\alpha \in \{f_1, \ldots, f_{k-1}, \Sigma^{-k+1}\sigma_Y\alpha\}$.*
3. *If $X \in \{f_1, \ldots, f_k\}$, then $F_k X \in \{f_1, \ldots, f_{k-1}\}$ and $\Sigma^{-1}(X/F_1 X) \in \{f_2, \ldots, f_k\}$.*

Proof. The first two statements follow by setting $n = 1$ and $m = 1$ respectively in [1, Proposition 2.3]. The third statement follows from [1, Proposition 2.2] and its analogue. $\square$

Often definitions of Toda brackets require various vanishing conditions. The following proposition shows that these are implicit in the above definition.

Proposition A.5. *There is an object X in $\{f_1, \ldots, f_{n-1}\}$ if and only if $0 \in \langle f_1, \ldots, f_{n-1} \rangle_T$.*

Proof. First assume $X \in \{f_1, \ldots, f_{n-1}\}$. By Lemma A.4, $\Sigma^{-1}(F_{n-1}X/F_1 X) \in \{f_2, \ldots, f_{n-2}\}$. Let β be the composite $\Sigma^{n-3}A_{n-1} \cong \Sigma^{-2}(F_n X/F_{n-1}X) \longrightarrow \Sigma^{-1}F_{n-1}X \longrightarrow \Sigma^{-1}(F_{n-1}X/F_1 X)$. Let α be the map $\Sigma^{-1}(F_{n-1}X/F_1 X) \longrightarrow F_1 X \cong A_0$. Since α is the cofiber of the map $\Sigma^{-1}F_{n-1}X \longrightarrow \Sigma^{-1}(F_{n-1}X/F_1 X)$, the composite $\alpha\beta$ is trivial. So $\alpha\beta = 0 \in \langle f_1, \ldots, f_{n-1} \rangle_T$.

For the other direction, assume $\theta = 0 \in \langle f_1, \ldots, f_{n-1} \rangle_T$. Then $\theta = \alpha\beta$ with $\beta \colon \Sigma^{n-3}A_{n-1} \longrightarrow Z$ and $\alpha \colon Z \longrightarrow A_0$ with $Z \in \{f_2, \ldots, f_{n-2}\}$. Since $\alpha\beta$ is trivial, taking cofibers gives a map $\gamma \colon \Sigma^{n-2}A_{n-1} \longrightarrow C_\alpha$. By Lemma A.4, $C_\alpha \in \{f_1, \ldots, f_{n-2}\}$ and $C_\gamma \in \{f_1, \ldots, f_{n-1}\}$. $\square$

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