Rigidity and algebraic models for rational equivariant stable homotopy theory

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Main Question

Question: Given stable model categories $\mathcal{C}$ and $\mathcal{D}$, if $\mathcal{H}o(\mathcal{C})$ and $\mathcal{H}o(\mathcal{D})$ are triangulated equivalent, does that imply that they are Quillen equivalent?

$$\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{H}o(\mathcal{D}) \overset{?}{\Rightarrow} \mathcal{C} \simeq_{Q} \mathcal{D}$$

Answer: Sometimes.

If for a fixed $\mathcal{D}$ the answer is yes for every $\mathcal{C}$, then we say $\mathcal{H}o(\mathcal{D})$ is *rigid*.

If the answer is no, we say the equivalence is *exotic*.

$\mathcal{D}(R)$ is rigid

Let Ch(R) be the model category of unbounded chain complexes of modules over a ring R. Then $\mathcal{H}o(\mathrm{Ch}(R)) = \mathcal{D}(R)$

Theorem: (Schwede - S. 2003) $\mathcal{D}(R)$ is rigid. That is, if $\mathcal{C}$ is a (nice) stable model category such that $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(R)$ then $\mathcal{C} \simeq_{Q}$ Ch(R). (Also holds for R a ring with many objects.)

Definition: $\mathcal{C}$ is *nice* if $\mathcal{C}$ is Quillen equivalent to a model category which is compatibly enriched over spectra. (For example, $\mathcal{C}$ is nice if it is cofibrantly generated, proper, simplicial, stable model category.)

Key to Proof: The Eilenberg-Mac Lane spectrum HR is determined by its homotopy.

$\mathcal{D}(R)$ is rigid

Theorem: (Schwede - S. 2003) $\mathcal{D}(R)$ is rigid. That is, if $\mathcal{C}$ is a (nice) stable model category such that $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(R)$ then $\mathcal{C} \simeq_Q$ Ch(R).

Proof:

- Given $F : \mathcal{D}(R) \xrightarrow{\simeq} \mathcal{H}o(\mathcal{C})$, let $F(R) = X$.
- $\pi_* \mathrm{End}_{\mathcal{C}}(X) \cong [X, X]_*^{\mathcal{H}o(\mathcal{C})} \cong [R, R]_*^{\mathcal{D}(R)} \cong R$
- $\mathrm{End}(X) \simeq HR$

$$\mathcal{C} \simeq_Q \mathrm{End}(X)\text{-mod} \simeq_Q HR\text{-mod} \simeq_Q \mathrm{Ch}(R)$$

Examples

Corollary: The homotopy category of rational, G-equivariant spectra for G *finite* is rigid.

Proof: $\{\Sigma^\infty_{\mathbb{Q}} G/H_+\}$ is a set of compact generators.
For G finite, have a single compact generator,

$$X = \bigvee \Sigma^\infty_{\mathbb{Q}} G/H_+$$

$\pi_n \mathrm{End}(X) = 0$ for $n > 0$.

Let $\pi_0 \mathrm{End}(X) = R$, then

$$\mathbb{Q} - G\text{-spectra} \simeq_Q \mathrm{End}(X)\text{-mod} \simeq_Q HR\text{-mod}$$

Examples

Corollary: (Barnes-Roitzheim 2012) The homotopy category of rational, G-equivariant spectra for G *profinite* is rigid.

Note: A set of compact generators is $\{\Sigma^\infty G/H_+\}$ for H the open subgroups of G for G profinite. Need the many objects, or ringoid, version here.

Non-Example, an exotic equivalence: (Dugger-S. 2009, Schlichting 2002) There are two DGAs A and B such that $\mathcal{D}(A) \simeq_\Delta \mathcal{D}(B)$ but the underlying model categories of differential graded A-modules and B-modules are not Quillen equivalent.

Differential Graded Algebras

Definition. A DGA is *formal* if it is quasi-isomorphic to its homology ring. That is, $A \simeq H_* A$.

Definition. A DGA is *intrinsically formal* if any DGA B with $H_* A \cong H_* B$ is quasi-isomorphic to A. That is, A is intrinsically formal if

$$H_* A \cong H_* B \text{ implies } A \simeq B.$$

Theorem (Additive Rigidity): If A is an intrinsically formal DGA and $\mathcal{C}$ is a nice, *additive*, stable model category such that $\mathcal{H}o(\mathcal{C}) \simeq_\triangle \mathcal{D}(A)$ then $\mathcal{C} \simeq_Q$ d.g. A-Mod. That is, $\mathcal{D}(A)$ is rigid among all nice, additive, stable model categories.

Proof:

Theorem (Additive Rigidity): If A is an intrinsically formal DGA and $\mathcal{C}$ is a nice, *additive*, stable model category such that $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(A)$ then $\mathcal{C} \simeq_Q$ d.g. A-Mod.

Proof:

- Given $F : \mathcal{D}(A) \xrightarrow{\simeq} \mathcal{H}o(\mathcal{C})$, let $F(A) = X$.
- $\mathrm{End}_{\mathcal{C}}(X)$ is a DGA.
- $H_* \mathrm{End}_{\mathcal{C}}(X) \cong [X, X]_*^{\mathcal{H}o(\mathcal{C})} \cong [A, A]_*^{\mathcal{D}(A)} \cong H_*(A)$
- $\mathrm{End}_{\mathcal{C}}(X) \simeq A$

$$\mathcal{C} \simeq_Q \text{d.g. } \mathrm{End}(X)\text{-mod} \simeq_Q \text{d.g. } A\text{-mod}$$

Rational Differential Graded Algebras

Theorem (Rational Rigidity): If A is an intrinsically formal, *rational* DGA and $\mathcal{C}$ is a nice stable model category such that $\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{D}(A)$ then $\mathcal{C} \simeq_{Q}$ d.g. A-Mod. That is, $\mathcal{D}(A)$ is rigid among all nice stable model categories.

Key to proof: Any rational ring spectrum is modeled by a rational DGA. (S. 2007)

Example: $H_* A \cong \mathbb{Q}[c]$

Additive Rigidity and Rational Rigidity results can also be formulated for $\mathcal{C}$ with a *set* of compact generators using DG-categories.

Commutative Differential Graded Algebras

Definition. A commutative DGA A is *intrinsically formal as a commutative DGA* if any other commutative DGA B with $H_*A \cong H_*B$ is quasi-isomorphic to A.

Examples.

- Any free commutative graded algebra is intrinsically formal.
- Products preserve intrinsic formality.
- Localization preserves intrinsic formality.
- $\mathbb{Q}, \quad \mathbb{Q}[c], \quad \mathbb{Q}[c, c^{-1}], \quad \prod_i \mathbb{Q}[c_i], \quad \mathcal{S}^{-1} \prod_i \mathbb{Q}[c_i]$
- $\mathbb{Q}[c] \longrightarrow \mathbb{Q}[c, c^{-1}] \longleftarrow \mathbb{Q}$

$\mathbb{Q}$ - T - equivariant spectra for T the circle group

$$\mathcal{R}_a^T = \{ \ \prod_F \mathbb{Q}[c_F] \longrightarrow \mathcal{E}^{-1} \prod_F \mathbb{Q}[c_F] \longleftarrow \mathbb{Q} \ \}$$

Here, $\mathbb{Q}[c_F] \cong H^*BT/F$ and $\mathcal{E} = \{c(V) | V^T = 0\}$

Proposition: Assume given a diagram of commutative DGAs $\mathcal{R}$ such that $H_*(\mathcal{R}) \cong \mathcal{R}_a^T$. Since $\mathcal{R}_a^T$ is intrinsically formal, $\mathcal{R}$ and $\mathcal{R}_a^T$ are quasi-isomorphic and hence,

$$\mathcal{R} \text{ - mod} \simeq_Q \mathcal{R}_a^T \text{ - mod}$$

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$$\mathcal{R} \text{ - mod} \simeq_Q \mathcal{R}_a^T \text{ - mod}$$

Theorem: (Greenlees - S. 2011) The model category of rational T-equivariant spectra is Quillen equivalent to the cellularization of modules over a diagram of commutative rings $\mathcal{R}_t^T$ such that $H_*(\mathcal{R}_t^T) \cong \mathcal{R}_a^T$. It follows that

$$\mathbb{Q} \text{ - } T \text{ - spectra} \simeq_Q \text{cell - } \mathcal{R}_t^T \text{ - mod} \simeq_Q \text{cell - } \mathcal{R}_a^T \text{ - mod}$$

$\mathbb{Q}$ - T^n - equivariant spectra (over a rank n torus)

Theorem: (Greenlees - S. 2011) The model category of rational T^n-equivariant spectra is Quillen equivalent to the cellularization of modules over an intrinsically formal diagram of commutative rings $\mathcal{R}_t^{T^n}$ such that $H_*(\mathcal{R}_t^{T^n}) \cong \mathcal{R}_a^{T^n}$. (Here, $\mathcal{R}_a^{T^n}$ is an explicit diagram of localizations and products of polynomial algebras.) It follows that

$$\mathbb{Q} \text{ - } T^n \text{ - spectra} \simeq_Q \text{cell - } \mathcal{R}_t^{T^n} \text{ - mod} \simeq_Q \text{cell - } \mathcal{R}_a^{T^n} \text{ - mod.}$$

Moreover,

$$\text{cell - } \mathcal{R}_a^{T^n} \text{ - mod} \simeq_Q \mathcal{A}(T^n) \text{ - mod}$$

here $\mathcal{A}(T^n)$ is an explicit abelian category, injective dimension n.

Rigidity of $\mathbb{Q}$ - T - equivariant spectra

Theorem: (Barnes and Roitzheim 2012) Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of $\mathbb{Q} - T$-equivariant spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of $\mathbb{Q} - T-$ equivariant spectra.

$$\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{H}o(\mathbb{Q} - T - \text{-quivariant spectra}) \quad \Rightarrow$$

$$\mathcal{C} \simeq_{Q} \mathbb{Q} - T - \text{equivariant spectra}$$

Proof: There is only one DG-category up to quasi-isomorphism with the homology ring and Massey products as described for $\mathcal{E}$. This information is determined by the triangulated structure on the homotopy category.

Endomorphism DGA for $\mathbb{Q}$ - T - equivariant spectra

Proposition: (S. 2002)
$\mathbb{Q}$ - T - equivariant spectra $\simeq_Q \mathcal{A}(T)$ - mod $\simeq_Q \mathcal{E}$ - mod
for $\mathcal{E}$ an endomorphism DG-category. (Here H, K are finite subgroups of T):

- $H_*\mathcal{E}(H, T) = \mathbb{Q}[0]$ generated by $[f^H]$,
- $H_*\mathcal{E}(T, H) = \mathbb{Q}[1]$ generated by $[g^H]$,
- $H_*\mathcal{E}(H, K) = 0$,
- $H_*\mathcal{E}(H, H) = \mathbb{Q}[0] \oplus \mathbb{Q}[1]$ gen'd by $[id_H]$ and $[m^H = g^H f^H]$,
- $H_*\mathcal{E}(T, T) = (\oplus_{n \geq 0} \mathbb{Q}\mathcal{F}[2n+1]) \oplus \mathbb{Q}[0]$
  generated by $[i^H_{2n+1}]$ and$[id_T]$;
  $\mathbb{Q}\mathcal{F}$ has basis $\mathcal{F}$, the set of finite subgroups of T.

The nontrivial products and Massey products are

- $[g^H][f^H] = [m^H]$
- $[f^H][g^H] = [i^H_1]$
- $\langle [f^H], [m^H], ..., [m^H], [g^H] \rangle = [-i^H_{2n+1}]$
  (where$[m^H]$ occurs n times).

Stable homotopy category of spectra

Theorem: (Schwede 2007) Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of spectra.

$$\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{H}o(\mathsf{Spectra}) \quad \Rightarrow \quad \mathcal{C} \simeq_{Q} \mathsf{Spectra}$$

2 - local G-equivariant spectra, for G finite

Theorem:(Patchkoria 2013) Let G be a finite group and $\mathcal{C}$ a cofibrantly generated, proper, G-equivariant stable model category. Suppose that $\Phi : \mathcal{H}o(G\text{- spectra}_{(2)}) \xrightarrow{\simeq_\Delta} \mathcal{H}o(\mathcal{C})$ is an equivalence of triangulated categories such that

$$\Phi(\Sigma^\infty_+ G/H) \simeq G/H_+ \wedge \Phi(\mathbb{S}_{(2)})$$

for any $H \leq G$ such that these isomorphisms are natural with respect to the restrictions, conjugations and transfers. Then there exists a Quillen equivalence

$$G\text{- spectra}_{(2)} \simeq_Q \mathcal{C}.$$

$K_{(2)}$-local spectra

Theorem: (Roitzheim 2007) Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of K-local spectra at the prime 2 are equivalent as triangulated categories, then they are Quillen equivalent.

$$\mathcal{H}o(\mathcal{C}) \simeq_{\triangle} \mathcal{H}o(K_{(2)}\text{-local Spectra}) \quad \Rightarrow \quad \mathcal{C} \simeq_{Q} K_{(2)}\text{-local Spectra}$$

Exotic Examples: (Bousfield 1985, Franke 1996, Roitzheim 2008) At an odd prime the homotopy category of K-local spectra is equivalent to the derived category of an abelian category.

$$\mathcal{H}o(K_{(p)}\text{-local Spectra}) \simeq \mathcal{D}(\mathbb{A})$$

It is not known if this is a triangulated equivalence.
The underlying model categories are *not* Quillen equivalent.

Exotic Examples

For E a ring spectrum, say $\pi_* E$ is *n sparse* if it is concentrated in dimensions divisible by a natural number n.

Let $\dim(\pi_* E)$ be the graded global homological dimension of $\pi_* E$.

Theorem: (Patchkoria 2011) Suppose that $\dim(\pi_* E)$ is d and $\pi_* E$ is n-sparse. If

- $d = 0$, or
- $d = 1$ and $n \geq 2$, or
- $d = 2$ and $n \geq 4$,

then there is an equivalence of categories:

$$\mathcal{H}o(E\text{-mod}) \simeq \mathcal{D}(\pi_* E).$$

Note: These equivalences are not known to be triangulated equivalences for $d = 1, 2$.

Exotic Examples

Corollary: Examples where $\mathcal{H}o(E\text{-mod}) \simeq \mathcal{D}(\pi_* E)$, but there is no underlying Quillen equivalence:

Dimension 0:

- Morava K-theory: $\pi_* K(n) \cong \mathbb{F}_p[v_n, v_n^{-1}], \quad |v_n| = 2(p^n - 1)$

Dimension 1, N sparse for $N \geq 2$:

- Complex K-theory: $\pi_* KU \cong \mathbb{Z}[u, u^{-1}], \quad |u| = 2$
- Connective Morava K-theory:
  $\pi_* k(n) \cong \mathbb{F}_p[v_n], \quad |v_n| = 2(p^n - 1)$

Dimension 2, N sparse for $N \geq 4$:

- Real connective K-theory localized at an odd prime p:
  $\pi_* ko_{(p)} \cong \mathbb{Z}_{(p)}[\omega], \quad |\omega| = 4$
- Truncated Brown-Peterson spectrum $BP\langle 1 \rangle$, odd prime p:
  $\pi_* BP\langle 1 \rangle = \mathbb{Z}_{(p)}[v_1], \quad |v_1| = 2(p - 1)$
- Johnson-Wilson spectrum E(2), odd prime p:
  $pi_* E(2) \cong \mathbb{Z}_{(p)}[v_1, v_2, v_2^{-1}], \quad |v_1| = 2(p - 1), |v_2| = 2(p^2 - 1)$