Derived equivalences for rings and DGAs

(Joint work with D. Dugger)

Definition:

(1) A **differential graded algebra (DGA)** is a chain complex A with products $A_k \otimes A_l \longrightarrow A_{k+l}$ such that $d(ab) = (da)b + (-1)^{|a|}a(db)$.

(2) A **differential graded module** over A is a chain complex M with actions $A_k \otimes M_l \longrightarrow M_{k+l}$ such that $d(am) = (da)m + (-1)^{|a|}a(dm)$.

(3) Let ***A*-Mod** denote the category of dg-modules. (For a ring R, R-Mod $\cong \mathcal{C}h_R$)

(4) A **quasi-isomorphism** is a homomorphism which induces an isomorphism in homology.

(5) A-Mod is a model category; its homotopy category is triangulated equivalent to the **derived category** of A, $\mathcal{D}(A)$.

$$\mathcal{H}o(A\text{-Mod}) \cong_{\Delta} \mathcal{D}(A)$$

Question: What is the relationship between the following three conditions?

(1) A and B are *quasi-isomorphic* DGAs.

(2) A-Mod and B-Mod are *Quillen equivalent* as model categories.

(3) $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are equivalent as triangulated categories. (A and B are *derived equivalent.*)

Easy answers:

- (1) $\Longrightarrow$ (2) or
quasi-isomorphism $\Rightarrow$ Quillen equivalence:

Given a quasi-isomorphism $A \xrightarrow{f} B$ then extension of scalars induces a Quillen equivalence:

$$A\text{-Mod} \xrightarrow{-\otimes_A B} B\text{-Mod}$$

- (2) $\Longrightarrow$ (3) or
Quillen equivalence $\Rightarrow$ derived equivalence:

Follows from the definition of Quillen equivalence since $\mathcal{H}o(A\text{-Mod}) \cong \mathcal{D}(A)$.

Outline of talk:

- (2) $\not\Longrightarrow$ (1) or
Quillen equivalence $\not\Rightarrow$ quasi-isomorphism

- (1)' $\Longleftrightarrow$(2) or
topological equivalence $\Longleftrightarrow$ Quillen equivalence

- For **rings** (2) $\Longleftrightarrow$ (3) or
Quillen equivalence $\Longleftrightarrow$ derived equivalence

- For DGAs in general, (3) $\not\Longrightarrow$ (2) or
derived equivalence $\not\Rightarrow$ Quillen equivalence

Quillen equivalence $\not\Rightarrow$ quasi-isomorphism

- *Morita theory* provides many examples where A and B are Quillen equivalent but not quasi-isomorphic: e.g, R and its matrix ring $M_n(R)$.

  Rickard's generalization: *"Tilting theory"* for derived equivalences of rings. (More details later.)

- Example:
  $A = \mathbb{Z}[e_1]/(e^4)$ with $de = 2$ and $A' = H_*A$
  $= \Lambda_{\mathbb{Z}/2}(\alpha_2)$

  (Here I drew a representation of these two DGAs.)

  A and A' are *not* quasi-isomorphic,
  (although $H_*A \cong H_*A'$.)

  Later we'll show A-Mod $\simeq_{Quillen}$ A'-Mod
  (and thus also $\mathcal{D}(A) \cong_{\Delta} \mathcal{D}(A')$.)

Spectral Algebra:

- Let $H\mathbb{Z}$ denote the Eilenberg-Mac Lane spectrum. (It represents $H^*(-;\mathbb{Z})$.)

  $H\mathbb{Z}$ is a (commutative) ring spectrum

- For C in $\mathcal{C}h_{\mathbb{Z}} \longrightarrow$ associate HC an $H\mathbb{Z}$-module with $H_*C \cong \pi_*HC$

$$\mathcal{C}h_{\mathbb{Z}} \simeq_{Quillen} H\mathbb{Z}\text{-Mod}$$

- For A a DGA $\longrightarrow$ associate HA an $H\mathbb{Z}$-algebra (4 steps; "H" is complicated)

$$DGA \simeq_{Quillen} H\mathbb{Z}\text{-}Algebras$$

- For a DGA A:

$$A\text{-Mod} \simeq_{Quillen} HA\text{-Mod}$$

Topological equivalence:

Ring spectra = S-algebras, (S the sphere spectrum.)

The unit map $S \to H\mathbb{Z}$ induces forgetful functor $H\mathbb{Z}$-algebras $\to$ S-algebras.

(Here I drew a picture of the path components in the homotopy category of S-algebras. Some of these components contain more than one path component of $H\mathbb{Z}$-algebras. This is where there are topologically equivalent DGAs which are not quasi-isomorphic.)

Definition: Two DGAs A and B are **topologically equivalent** if HA and HB are weakly equivalent as ring spectra (S-algebras).

Quasi-isomorphism $\Rightarrow$ topological equivalence

Analogy in algebra: there are many pairs of non-isomorphic $\mathbb{Z}[x]$-algebras which are isomorphic as $\mathbb{Z}$-algebras (rings).

Example: $H\mathbb{Z} \wedge_S H\mathbb{Z}/2$

- Two $H\mathbb{Z}$-algebra structures:
  L and R (left and right $H\mathbb{Z}$ actions)
- Only one ring spectrum: $L = R = H\mathbb{Z} \wedge_S H\mathbb{Z}/2$.

To see $L \not\simeq R$ as $H\mathbb{Z}$-algebras (or DGAs) compute:

$$H_*(L \wedge_{H\mathbb{Z}} H\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

$$H_*(R \wedge_{H\mathbb{Z}} H\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}_2[\xi_1^2, \xi_2, \ldots][\epsilon]/\epsilon^2 \otimes \mathbb{F}_2[\xi_1, \xi_2, \ldots].$$

L and R are big DGAs; it is hard to write down explicit models for them.

The same sort of analysis works with $H\mathbb{Z} \wedge_{bo} H\mathbb{Z}/2$ but the DGAs are finite.

Revisit above example:
$A = \mathbb{Z}[e_1]/(e^4), de = 2$, and $A' = H_*A = \Lambda_{\mathbb{Z}/2}(\alpha_2)$ are topologically equivalent.

- Claim: $HA \simeq HA'$ as ring spectra, (although $A \not\simeq A'$ as DGAs.)

- Use HH^* and THH^*:

  For a ring R and an R-bimodule M, DGAs with non-zero homology $H_0 = R$ and $H_n = M$ are classified by $HH^{n+2}_{\mathbb{Z}}(R; M) = THH^{n+2}_{H\mathbb{Z}}(HR; HM)$

  Ring spectra are classified by $THH^{n+2}_S(HR; HM)$.

- Compute: For $R = \mathbb{Z}/2$, $M = \mathbb{Z}/2$:

$$HH^*_{\mathbb{Z}}(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[\sigma_2]$$

  (Franjou, Lannes, and Schwartz)

$$THH^*_S(\mathbb{Z}/2; \mathbb{Z}/2) = \Gamma_{\mathbb{Z}/2}[\tau_2]$$

- $S \to H\mathbb{Z}$ induces $\Phi : HH^*_{\mathbb{Z}}(\mathbb{Z}/2) \to THH^*_S(\mathbb{Z}/2)$.

  For $n = 0$, in HH^2 we have $\sigma \leftrightarrow \mathbb{Z}/4$ and $0 \leftrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$. So $\Phi(\sigma) = \tau$.

  For $n = 2$, in HH^4 we have $\sigma^2 \leftrightarrow A$ and $0 \leftrightarrow A'$. $\Phi(\sigma^2) = \Phi(0) = 0$, so $HA \simeq HA'$ as ring spectra.

Proposition.
If A and B are topologically equivalent DGAs then A and B are Quillen equivalent (and also derived equivalent).

Proof. Given a weak equivalence of ring spectra $HA \xrightarrow{f} HB$, then extension of scalars induces a Quillen equivalence

$$HA\text{-Mod} \xrightarrow{-\wedge_{HA} HB} HB\text{-Mod}$$

Since $A\text{-Mod} \simeq_{Quillen} HA\text{-Mod}$ and
$HB\text{-Mod} \simeq_{Quillen} B\text{-Mod}$ it follows that

$$A\text{-Mod} \simeq_{Quillen} B\text{-Mod}\,.$$

□

Theorem
Let A and B be two DGAs. There is a Quillen equivalence $F : A\text{-Mod} \to B\text{-Mod}$ such that $F(A) \simeq B$ if and only if A and B are topologically equivalent.

Fixing $F(A) \simeq B$ is a way to exclude Morita equivalences. So, this theorem says Quillen equivalences between DGAs only arise from Morita equivalences and topological equivalences.

Proposition.
Let k be a commutative ring. Suppose A and B are augmented differential graded k-algebras, then A and B are quasi-isomorphic if and only if they are topologically equivalent.

Proof. Use a change-of-rings long exact sequence for THH for $S \to H\mathbb{Z} \to Hk$. The augmentation gives a splitting $(Hk \to HA \to Hk)$ which implies that the map $\Phi : HH^* = THH^*_{H\mathbb{Z}} \to THH^*_S$ is injective. □

This shows that in many situations topological equivalences do not provide anything new.

Derived equivalence ⇒ Quillen equivalence for rings

Consider **rings** instead of DGAs.

Definition: An $R - R'$ **tilting complex** T is a chain complex of finitely generated projective R-modules such that

(1) T is a **generator**; that is, if $[T, X]_*^{\mathcal{D}(R)} = 0$ then $X = 0$ in $\mathcal{D}(R)$.

(2) $[T, T]_0^{\mathcal{D}(R)} \cong R'$ and $[T, T]_n^{\mathcal{D}(R)} = 0$ whenever $n \neq 0$.

Theorem. (Rickard)
R is derived equivalent to R' if and only if there exists an $R - R'$ tilting complex.

Corollary. (Schwede - S.; Dugger - S.)
The following are equivalent:
(1) R and R' are derived equivalent.
(2) $\mathcal{C}h_R$ and $\mathcal{C}h_{R'}$ are Quillen equivalent.
(3) There exists an $R - R'$ tilting complex.

Key: a DGA with homology concentrated in degree zero is determined by its homology.

Proposition.
If $\mathcal{C}h_R$ and $\mathcal{C}h_{R'}$ are Quillen equivalent, then their algebraic K-groups are isomorphic:
$K(R) \cong K(R')$.

Waldhausen and Thomason have similar statements, but require more hypotheses.

Putting together the last two statements gives:

Corollary.
If R and R' are derived equivalent then
$K(R) \cong K(R')$.

Neeman proved this result for *regular rings*.

We have results for derived equivalences of abelian categories as well. Neeman's results for abelian categories are even more general though.

Our proofs are much simpler; we do not construct K-theory from the derived category.

Derived equivalence $\not\Rightarrow$ Quillen equivalence for DGAs

Proposition.
There are DGAs B and $B'(= H_*B)$ such that B-Mod *and* B'-Mod are not *Quillen equivalent even though B and B'* are *derived equivalent.*

$$\mathcal{D}(B) \cong_\Delta \mathcal{D}(B') \text{ but } B\text{-Mod} \not\simeq_{Quillen} B'\text{-Mod}$$

- Example due to M. Schlichting.

Let $R = \mathbb{Z}/p^2$ and $R' = \mathbb{Z}/p[\epsilon]/[\epsilon^2]$ $(p \neq 2)$.

Let $\mathcal{M}(R)$ and $\mathcal{M}(R')$ be the stable model categories. (Stable equivalences are 'isomorphisms modulo projectives', cofibrations are injections and fibrations are surjections.)

One can show:
$$\mathcal{H}o(\mathcal{M}(R)) \cong_\Delta \mathbb{Z}/p\text{-vector spaces} \cong_\Delta \mathcal{H}o(\mathcal{M}(R'))$$

These cannot be Quillen equivalent though, because the Waldhausen K_4 groups are different.

$$\mathcal{M}(R) \not\simeq_{Quillen} \mathcal{M}(R')$$

Proof of proposition: There are DGAs associated to Schlichting's example so that:
$\mathcal{M}(R) \simeq_{Quillen} B$-Mod and
$\mathcal{M}(R') \simeq_{Quillen} B'$-Mod.

Thus, from Schlichting's work,

$$\mathcal{D}(B) \cong_{\Delta} \mathcal{D}(B') \text{ but } B\text{-Mod} \not\simeq_{Quillen} B'\text{-Mod}.$$

Keller also developed such DGAs, but only considers derived equivalences, $\mathcal{D}(B) \cong \mathcal{H}o(\mathcal{M}(R))$.

Independently we also show that B and B' are not topologically equivalent by considering HH^* and THH^* of their second Postnikov sections. This analysis is simpler than calculating K_4. □

John Rognes pointed out after the talk that there may be a mistake in the K_4 calculation quoted by Schlichting.