Solutions for Math 330 HW4

From Chapter 3:

12. Suppose that H is a proper subgroup of Z under addition and H contains 18, 30 and 40. Determine H.

Answer: Since 18 and 30 are in H, so are their inverses -18 and -30 since subgroups are closed under inverses. Since subgroups are closed under the operation, this means H contains (-18) + 30 = 12 and (-30) + 40 = 10. By closure of inverses, H then contains -10 and by closure under the operation H also contains (-10) + 12 = 2.

Since H contains 2, it must contain all integral multiples of 2 (all evens). If H contained any other number k (an odd number), then H would also contain k + (-k + 1) = 1 (since -k + 1 would be even). If H contains 1 then it would contain all integers and not be a proper subgroup. Thus, H is exactly the subgroup generated by 2, $< 2 >$, or the evens.

20. If H is a subgroup of G, then by the centralizer C(H) of H we mean the set $\{x \in G | xh = hx$ for all $h \in H\}$. Prove that C(H) is a subgroup of G.

Answer: Use the two step subgroup test. First, C(H) is nonempty: The identity element in G, e, is in C(H) because eh = h = he for all $h \in H$. Second C(H) is closed under inverses: Assume x is in C(H) and the inverse of x is y so $xy = e = yx$. We're given $xh = hx$ for all $h \in H$. Multiply this equation by y on both sides to get $yxhy = yhxy$. Since $xy = e = yx$ this simplifies to give $ehy = yhe$ which implies $hy = yh$. This holds for all $h \in H$, so y is also in C(H). Third, C(H) is closed under the operation: assume a and b are in C(H) we want to show ab is in C(H). We're given that $ah = ha$ and $hb = bh$ for all $h \in H$. Then $abh = a(bh) = a(hb)$ since $b \in C(H)$ and $a(hb) = (ah)b = (ha)b = hab$ since $a \in C(H)$. Thus we conclude $abh = hab$ for all $h \in H$ and $ab \in C(H)$.

From Chapter 4:

2. Suppose that $< a >, < b >, < c >$ are cyclic groups of order 6, 8 and 20, respectively. Find all generators of $< a >, < b >, < c >$.

Answer: Use Corollary 2, p. 77. The generators of $< a >$ are a and a^5. The generators of $< b >$ are b, b^3, b^5, b^7. The generators of $< c >$ are c, c^3, c^7, c^9, c^{11}, c^{13}, c^{17} and c^{19}. (Compare to Z_6, Z_8 and Z_{20}.)

8. Compute the orders of the following elements. Answer: Use the formula $n/\gcd(n, k)$. a) all order 5, b) all order 3, c) all order 15.

14. Suppose that a cyclic group G has exactly three subgroups: G, $\{e\}$ and a subgroup of order 7. What is the order of G, $|G|$? What can you say if 7 is replaced with p where p is a prime?

Answer: Since G is cyclic it has a subgroup for each divisor of $|G|$. We want a number n whose only three divisors are 1, 7 and n. Since 7 is a divisor of n, then $n/7$ is also a divisor. So $n/7$ must be 1 or 7. It must be 7 (if $n = 7$ then there are only two subgroups). Thus $n = 49$, so $|G| = 49$. If we replace 7 by p then $|G| = p^2$.

48. If G is a cyclic group and 15 divides the order of G, determine the number of solutions in G of the equation $x^{15} = e$. If 20 divides the order of G, determine the number of solutions of $x^{20} = e$. Generalize.

Answer: 15, 20: Since 15 divides the order of the cyclic group G, there is exactly one subgroup H of order 15. (For example H could be Z_{15}.) Each of the elements in H has an order which divides 15, so each element satisfies $x^{15} = e$. So there are 15 elements in G such that $x^{15} = e$. Similarly there are 20 in the second case. In general, when n divides the order of a cyclic group, the number of solutions of $x^n = e$ is n.