

# CONVERGENCE OF THE HOMOLOGY SPECTRAL SEQUENCE OF A COSIMPLICIAL SPACE

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**1. Introduction.** In this paper we study the convergence properties of the homology spectral sequence of a cosimplicial space. The Eilenberg-Moore spectral sequence is an example of this spectral sequence applied to the cobar construction of a fibre square. Hence this spectral sequence is also called the generalized Eilenberg-Moore spectral sequence.

This work builds on results due to W. Dwyer and A. K. Bousfield. W. Dwyer considered the convergence properties of the Eilenberg-Moore spectral sequence for a fibration in [D1]. Then A. K. Bousfield used these results as a basis for finding convergence conditions for the generalized Eilenberg-Moore spectral sequence [B2]. We continue in this direction. In section 3 we consider new convergence conditions for the Eilenberg-Moore spectral sequence of a fibre square. Using these results and adding a finite type assumption we obtain a new pro-convergence result, Theorem 5.3. Combining this result with Corollary 1.2 from [S] gives conditions for strong convergence (Theorem 6.1) which replace Bousfield's one-connectedness requirement in [B2, 3.6] by a  $p$ -good requirement.

In [D2] W. Dwyer analyzed what the Eilenberg-Moore spectral sequence for a fibration is converging to when it is not necessarily converging to the homology of the fibre. We consider this exotic convergence question in the case of the generalized Eilenberg-Moore spectral sequence in section 7. In section 8 we note that our exotic convergence result for the Eilenberg-Moore spectral sequence of a fibre square generalizes one of W. Dwyer's results for a fibration.

In section 2 we construct the homology spectral sequence of a cosimplicial space. We also define pro-convergence and strong convergence. Section 3 contains the various Eilenberg-Moore spectral sequence convergence results. The special case of pro-convergence for a cosimplicial simplicial abelian group is considered in section 4. As mentioned above, sections 5 and 6 contain the main pro-convergence and strong convergence results: Theorem 5.3 and Theorem 6.1. The homology of mapping spaces is discussed at the end of section 6 as an application of Theorem 6.1. Section 7 considers exotic convergence results. In section 8

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we apply the exotic convergence results from section 7 to the Eilenberg-Moore spectral sequence. Relative convergence is considered in section 9. In the last section we consider the total space of the  $p$ -resolution of a cosimplicial space. We also prove another strong convergence theorem which generalizes Theorem 6.1.

This paper is written simplicially, so “space” means “simplicial set.” Throughout, the homology of spaces is homology with coefficients in the field with  $p$  elements. Let  $R = \mathbb{F}_p$ .

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**2. The homology spectral sequence for a cosimplicial space.** In this section we recall some of the necessary definitions for cosimplicial spaces and towers. Then we construct the homology spectral sequence for a cosimplicial space  $X'$ . Finally, we consider definitions of strong convergence and pro-convergence.

The main objects of study in this paper are cosimplicial spaces. We use the model category structure on cosimplicial spaces developed in [BK, X]. A map  $f : X' \rightarrow Y'$  is a *weak equivalence* if  $f^n : X^n \rightarrow Y^n$  is a weak equivalence for  $n \geq 0$ . The map  $f$  is a *fibration* if

$$X^n \rightarrow Y^n \times_{M^{n-1}Y'} M^{n-1}X'$$

is a fibration for  $n \geq 0$ . Here  $M^n X' = \{(x^0, \dots, x^n) \in X^n \times \dots \times X^n \mid s^i x^j = s^{j-1} x^i \text{ for } 0 \leq i < j \leq n\}$  for  $n \geq 0$  and  $M^{-1} = *$ . The map  $s : X^n \rightarrow M^{n-1}X'$  is induced by  $s^0 \times \dots \times s^n$ . The map  $f$  is a *cofibration* if  $f^n : X^n \rightarrow Y^n$  is a cofibration of simplicial sets for  $n \geq 0$  and  $f$  induces an isomorphism on the maximal augmentations. The maximal augmentation of  $X'$  is the subspace of  $X^0$  which consists of the simplices  $x \in X^0$  such that  $d^0 x = d^1 x$ .  $X'$  is *fibrant (cofibrant)* if  $X' \rightarrow * \rightarrow *$  is a fibration ( $\emptyset \rightarrow X'$  is a cofibration).

Let  $\Delta'$  be the cosimplicial space with  $\Delta^m = \Delta[m]$  the simplicial  $m$ -simplex for  $m \geq 0$ . Let  $\text{Tot } X' = \text{Hom}(\Delta', X')$  and  $\text{Tot}_s X' = \text{Hom}(\Delta^{[s]}, X')$  where  $\Delta^{[s]}$  is the simplicial  $s$ -skeleton of  $\Delta'$ .

**LEMMA 2.1.** *A weak equivalence between fibrant cosimplicial spaces  $X' \xrightarrow{\cong} Y'$  induces a weak equivalence  $\text{Tot}_s X' \xrightarrow{\cong} \text{Tot}_s Y'$ .*

*Proof.* Since  $\Delta^{[s]}$  is cofibrant, this lemma follows from the simplicial model category structure on cosimplicial spaces. See [BK, X 5.2].  $\square$

This lemma shows that on fibrant cosimplicial spaces  $\text{Tot}_s$  is homotopy invariant. This is not true in general. For a nonfibrant cosimplicial space  $X$  we replace  $X$  by a weakly equivalent fibrant cosimplicial space  $\overline{X}$ . Since the choice of fibrant replacement is unique up to weak equivalence,  $\text{Tot}_s \overline{X}$  is homotopy invariant.

Towers are useful for studying convergence properties of spectral sequences. A good reference for towers is [BK, III]. Over any category a map of towers  $\{f_s\} : \{A_s\} \rightarrow \{B_s\}$  is a *pro-isomorphism* if for each  $s$  there is a  $t$  and a map from  $B_{s+t}$  to  $A_s$  which makes the following diagram commute.

$$\begin{array}{ccc} A_{s+t} & \xrightarrow{f_{s+t}} & B_{s+t} \\ \downarrow & \swarrow & \downarrow \\ A_s & \xrightarrow{f_s} & B_s \end{array}$$

A map of towers of spaces  $\{X_s\} \rightarrow \{Y_s\}$  is a *weak pro-homotopy equivalence* if it induces a pro-isomorphism of sets  $\{\pi_0 X_s\} \rightarrow \{\pi_0 Y_s\}$  and for each  $i$  and  $s$  there exists a  $t$  such that for each vertex  $v \in X_{s+t}$  there exists a homomorphism  $\pi_i(Y_{s+t}, v) \rightarrow \pi_i(X_s, v)$  making the following diagram commute.

$$\begin{array}{ccc} \pi_i(X_{s+t}, v) & \longrightarrow & \pi_i(Y_{s+t}, v) \\ \downarrow & \swarrow & \downarrow \\ \pi_i(X_s, v) & \longrightarrow & \pi_i(Y_s, v) \end{array}$$

For pointed towers of connected spaces this condition is equivalent to having a pro-isomorphism of sets on  $\pi_0$  and a pro-isomorphism of groups for each tower of higher homotopy groups. A weak pro-homotopy equivalence induces a *pro-homology isomorphism*, i.e.  $\{H_n X_s\} \rightarrow \{H_n Y_s\}$  is a pro-isomorphism for all  $n$  [B2, 8.5].

We now construct the mod  $p$  homology spectral sequence for a cosimplicial space,  $\{E^r(X)\}$ , which abuts to  $H_*(\text{Tot } X; \mathbb{F}_p)$ . See also [B2].

Let  $\text{sa}$  ( $\text{ca}$ ) be the category of (co)simplicial abelian groups. Let  $\text{csa}$  be the category of cosimplicial simplicial abelian groups. For  $G$  in  $\text{sa}$ , let  $N_*G$  be the normalized chain complex with  $N_n G = G_n / \text{im } s_0 + \dots + \text{im } s_{n-1}$  and boundary  $\partial = \Sigma(-1)^i d_i$ . For  $B$  in  $\text{ca}$ , let  $N^*B$  be the normalized cochain complex with  $N^n B = B^n \cap \ker s^0 \cap \dots \cap \ker s^{n-1}$  with boundary  $\delta = \Sigma(-1)^i d^i$ . For  $B$  in  $\text{csa}$  let  $N^*N_*B$  be the normalized double chain complex.

Given  $B'$  in CSA construct the total complex  $TB'$  with

$$(TB')_n = \prod_{m \geq 0} N^m N_{m+n} B', \quad \partial_T = \partial + (-1)^{n+1} \delta.$$

Let  $T_m B' = TB' / F^{m+1} TB'$  where  $(F^{m+1} TB')_n = \prod_{k \geq m+1} N^k N_{k+n} B'$ . Then  $TB' = \varprojlim T_m B'$ .

For a cosimplicial space  $X'$ , let  $R \otimes X'$  be the cosimplicial simplicial vector space generated by  $X'$ , i.e.,  $(R \otimes X')_n^m$  is the vector space generated by the set  $X_n^m$ . Let  $\{E^r(X')\}$  be the homology spectral sequence of the filtered chain complex  $T(R \otimes X')$ . In other words, the spectral sequence comes from the following exact couple.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_* T_s(R \otimes X') & \longrightarrow & H_* T_{s-1}(R \otimes X') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & & E_{s+1}^1(X') & & E_s^1(X') & & \cdots \end{array}$$

Hence we can identify  $E_{s,t}^1 = N^s H_t(X')$  and  $d^1 = \Sigma (-1)^i d^i$ . So  $E_{s,t}^2 = \pi^s H_t(X')$ .

The tower  $\{H_* T_s(R \otimes X')\}$  contains all of the information needed for convergence questions about this spectral sequence. Occasionally we need to translate between the usual spectral sequence language and the analogous tower theoretic descriptions. For instance, the following two lemmas give very useful translations.

LEMMA 2.2. *If for some  $r$  the map  $f : X' \rightarrow Y'$  induces an  $E^r$ -isomorphism between spectral sequences  $\{E^r(X')\} \rightarrow \{E^r(Y')\}$ , then  $f$  induces a pro-isomorphism of towers*

$$\{H_n T_s(R \otimes X')\} \rightarrow \{H_n T_s(R \otimes Y')\}$$

for each  $n$ .

*Proof.* Consider the  $r$ th-derived exact couple.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_* T_s^{(r)}(R \otimes X') & \longrightarrow & H_* T_{s-1}^{(r)}(R \otimes X') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & & E_{s+1}^r(X') & & E_s^r(X') & & \cdots \end{array}$$

Here  $H_* T_s^{(r)}(R \otimes X')$  is the image of  $i_* : H_* T_{s+r}(R \otimes X') \rightarrow H_* T_s(R \otimes X')$  in the tower  $\{H_* T_s(R \otimes X')\}$ . Note that  $H_* T_{-1}(R \otimes X') = 0$ . So  $H_* T_0^{(r)}(R \otimes X') \cong$

$E_0^r(X')$ . Hence by induction and the five-lemma, an  $E^r$ -isomorphism induces an isomorphism  $H_*T_s^{(r)}(R \otimes X') \rightarrow H_*T_s^{(r)}(R \otimes Y')$ . This is enough to show that  $f$  induces a pro-isomorphism of towers  $\{H_*T_s(R \otimes X')\} \rightarrow \{H_*T_s(R \otimes Y')\}$ .  $\square$

Any weak equivalence between cosimplicial spaces  $X' \rightarrow Y'$  induces an  $E^2$ -isomorphism,  $\pi^*H_*(X') \xrightarrow{\cong} \pi^*H_*(Y')$ . Hence by Lemma 2.2 a weak equivalence induces a pro-isomorphism  $\{H_*T_s(R \otimes X')\} \rightarrow \{H_*T_s(R \otimes Y')\}$ .

Before we state the next lemma we need two definitions. A tower which is pro-isomorphic to a constant tower is called *pro-constant*. A tower which is pro-isomorphic to the trivial constant tower is called *pro-trivial*.

LEMMA 2.3. [B2, 3.5] *For any integer  $n$ , the tower  $\{H_nT_s(R \otimes X')\}$  is pro-constant if and only if for each  $s$  there exists  $r < \infty$  with  $E_{s,s+n}^r(X') = E_{s,s+n}^\infty(X')$  and for each sufficiently large  $s$   $E_{s,s+n}^\infty(X') = 0$ . The tower is pro-trivial if and only if for each  $s$  there exists  $r < \infty$  with  $E_{s,s+n}^r(X') = 0$ .*

*Proof.* The proof is a straightforward translation between tower information and spectral sequence information.  $\square$

In [B2, 2.2] Bousfield writes down compatible maps

$$\phi_s : H_*(\text{Tot}_s X') \rightarrow H_*T_s(R \otimes X').$$

Using the map induced from  $\text{Tot} X' \rightarrow \text{Tot}_s X'$  we get the following tower maps.

$$\{H_* \text{Tot} X'\} \xrightarrow{P} \{H_* \text{Tot}_s X'\} \xrightarrow{\Phi} \{H_*T_s(R \otimes X')\}$$

A cosimplicial space  $X'$  is *strongly convergent* if

$$\Phi \circ P : \{H_n \text{Tot}_s \bar{X}'\} \rightarrow \{H_n T_s(R \otimes \bar{X}')\}$$

is a pro-isomorphism for each integer  $n$ . The homology spectral sequence  $\{E^r(X')\}$  is called strongly convergent if  $X'$  is strongly convergent.

Lemma 2.3 translates the above tower theoretic definition of strong convergence into the usual structural strong convergence statements. Note that  $\{E^r(X')\}$  is strongly convergent if and only if for each  $n$   $\{H_n T_s(R \otimes X')\}$  is pro-constant with constant value  $H_n \text{Tot} X'$ . Hence Lemma 2.3 shows that there are two necessary conditions for strong convergence. First, there must only be finitely many nonzero differentials emanating from and terminating at any one  $E_{s,t}$  and only finitely many nontrivial filtrations on any one total degree line by  $E^\infty$ . Second, the spectral sequence must eventually vanish in negative total degrees.

A cosimplicial space  $X'$  is *pro-convergent* if

$$\Phi : \{H_n \text{Tot}_s \bar{X}'\} \rightarrow \{H_n T_s(R \otimes \bar{X}')\}$$

is a pro-isomorphism for each integer  $n$ . If  $X'$  is pro-convergent then the homology spectral sequence  $\{E^r(X')\}$  is also called pro-convergent.

The following proposition, which is a generalization of [B2, 8.6], is used in the proofs of Theorem 4.1 and Theorem 5.3. First we need a definition. A tower of cosimplicial spaces  $\{X'_s\}$  will be called *pro-convergent* if

$$\{H_n \text{Tot}_s \bar{X}'_s\} \rightarrow \{H_n T_s(R \otimes \bar{X}'_s)\}$$

is a pro-isomorphism for each  $n$ .

**PROPOSITION 2.4.** *Let  $h : \{X'_s\} \rightarrow \{Y'_s\}$  be a map of towers of cosimplicial spaces such that  $h : \{X'_s\} \rightarrow \{Y'_s\}$  is a weak pro-homotopy equivalence for each  $m \geq 0$ . Then  $\{X'_s\}$  is pro-convergent if and only if  $\{Y'_s\}$  is pro-convergent.*

*Proof.*  $\{X'_s\}$  is pro-convergent if and only if  $\{\bar{X}'_s\}$  is pro-convergent. The map  $h$  induces a map  $\{\bar{X}'_s\} \rightarrow \{\bar{Y}'_s\}$  of fibrant replacements which is also a weak pro-homotopy equivalence on each codegree. Hence we reduce to the case where all of the cosimplicial spaces in the lemma are fibrant.

Because  $h$  induces weak pro-homotopy equivalences on each codegree, it induces a pro-homology isomorphism  $\{H_n(X'_s)\} \rightarrow \{H_n(Y'_s)\}$  for each  $m$  and  $n$ . So by the five lemma for pro-isomorphisms, [BK, III 2.7], it induces a pro-isomorphism  $\{\pi^m H_n X'_s\} \rightarrow \{\pi^m H_n Y'_s\}$  for each  $m$  and  $n$ . In other words, it induces a pro-isomorphism of each tower  $\{E_{m,n}^2(X'_s)\} \rightarrow \{E_{m,n}^2(Y'_s)\}$ .

Lemma 2.2 can be restated for a map of towers of cosimplicial spaces to show that a pro-isomorphism of  $E^2$  towers induces a pro-isomorphism of towers  $\{H_* T_s(R \otimes X'_s)\} \rightarrow \{H_* T_s(R \otimes Y'_s)\}$ . This new statement can be proved by following the proof of Lemma 2.2 using the five lemma for pro-isomorphisms in place of the usual five lemma. Thus we can conclude that the right-hand map in the following diagram is a pro-isomorphism for each  $n$ .

$$\begin{array}{ccc} \{H_n \text{Tot}_s X'_s\} & \longrightarrow & \{H_n T_s(R \otimes X'_s)\} \\ \downarrow & & \downarrow \\ \{H_n \text{Tot}_s Y'_s\} & \longrightarrow & \{H_n T_s(R \otimes Y'_s)\} \end{array}$$

If the left-hand map in this diagram is a pro-isomorphism then the top map will be a pro-isomorphism if and only if the bottom map is a pro-isomorphism. This is equivalent to the statement of the proposition. So to conclude the proof we only need to show that the left-hand map is a pro-isomorphism.

We proceed by induction to show that  $\{\text{Tot}_t X'_s\}_s \rightarrow \{\text{Tot}_t Y'_s\}_s$  is a weak pro-homotopy equivalence for each  $t$ . First,  $\text{Tot}_0 X' = X^0$ . Since  $h : \{X'_s\} \rightarrow \{Y'_s\}$  is a weak pro-homotopy equivalence,  $\text{Tot}_0 h : \{\text{Tot}_0 X'_s\} \rightarrow \{\text{Tot}_0 Y'_s\}$  is also.

Throughout this paper *fibre square* refers to a pull-back square where at least

one map is a fibration. For a fibrant cosimplicial space  $X'$ ,  $\text{Tot}_t X'$  can be built up inductively by fibre squares as follows. See also [B1, p.149-150].

$$\begin{array}{ccc} \text{Tot}_t X' & \longrightarrow & \text{Hom}(\Delta[t], X^t) \\ \downarrow & & \downarrow \\ \text{Tot}_{t-1} X' & \longrightarrow & P \end{array}$$

Here  $P$  is the pull-back of the following diagram.

$$\begin{array}{ccc} & & \text{Hom}(\Delta[t], M^{t-1} X') \\ & & \downarrow \\ \text{Hom}(\partial\Delta[t], X^t) & \longrightarrow & \text{Hom}(\partial\Delta[t], M^{t-1} X') \end{array}$$

For the induction step we assume  $\text{Tot}_{t-1} h : \{\text{Tot}_{t-1} X'_s\} \rightarrow \{\text{Tot}_{t-1} Y'_s\}$  is a weak pro-homotopy equivalence. Since  $h$  induces a weak pro-homotopy equivalence on each codegree, it induces a weak pro-homotopy equivalence on each of the corners of the diagram for building  $\text{Tot}_t h : \{\text{Tot}_t X'_s\}_s \rightarrow \{\text{Tot}_t Y'_s\}_s$  as a pull-back. Using the five lemma for pro-isomorphisms, [BK, III 2.7], one can show that a map between towers of fibre squares induces a weak pro-homotopy equivalence on the pull-back towers if it is a weak pro-homotopy equivalence on the other towers. Thus  $\text{Tot}_t h : \{\text{Tot}_t X'_s\}_s \rightarrow \{\text{Tot}_t Y'_s\}_s$  is a weak pro-homotopy equivalence.

So by induction  $\{\text{Tot}_t X'_s\} \rightarrow \{\text{Tot}_t Y'_s\}$  is a weak pro-homotopy equivalence for each  $t$ . Hence, by considering the diagonal,  $\{\text{Tot}_s X'_s\} \rightarrow \{\text{Tot}_s Y'_s\}$  is a weak pro-homotopy equivalence. So it induces a pro-isomorphism on homology.  $\square$

The following corollary states the specific case which will be used in the proofs of Theorem 4.1 and Theorem 5.3.

**COROLLARY 2.5.** *Let  $h : \{X'_s\} \rightarrow \{Y'_s\}$  be a map of towers of cosimplicial spaces such that  $h : \{X^m\} \rightarrow \{Y^m\}$  is a weak pro-homotopy equivalence for each  $m \geq 0$ . If each  $Y'_s$  is pro-convergent, then so is  $X'$ .*

*Proof.* From the definition of pro-convergence for towers we can see that a tower of pro-convergent cosimplicial spaces is a pro-convergent tower. Also, for a constant tower, pro-convergence for the tower is equivalent to pro-convergence for the cosimplicial space itself. Thus this corollary follows easily from Proposition 2.4.  $\square$

The following lemma shows one way of ensuring that each homotopy group of the partial total spaces is finite. This will be used in the proof of Lemma 5.4.

**LEMMA 2.6.** *Let  $X'$  be a cosimplicial space. If  $\pi_m(X^n, *)$  is finite for each  $m$ ,  $n$  and choice of base point in  $X^n$ , then  $\pi_m(\text{Tot}_s \overline{X}', *)$  is finite for each  $m$ ,  $s$  and choice of base point in  $\text{Tot}_s \overline{X}'$ .*

*Proof.* Since  $X'$  is weakly equivalent to  $\overline{X}'$  we see that  $\pi_m(\overline{X}^n, *)$  is also finite for each  $m$ ,  $n$ , and choice of base point. Hence we can assume that  $X'$  is a fibrant cosimplicial space. Note that  $\text{Tot}_0 X'$  is  $X^0$ . So  $\pi_m(\text{Tot}_0 X', *)$  is finite for each  $m \geq 0$  and for each choice of base point. We proceed by induction. Assume that  $\pi_m(\text{Tot}_{k-1} X', *)$  is finite for each  $m \geq 0$  and for each base point. Consider the fibration  $p_k : \text{Tot}_k X' \rightarrow \text{Tot}_{k-1} X'$ . We must consider each of the components separately. Consider a vertex  $b \in \text{Tot}_k X'$ . Let  $F_b$  be the fibre of  $p_k$  over  $p_k(b) \in \text{Tot}_{k-1} X'$ . Let  $b_0$  be the vertex in  $X^0$  which is in the image of  $b$ . By using the construction discussed in the proof of Lemma 2.4 of the fibration  $p_k : \text{Tot}_k X' \rightarrow \text{Tot}_{k-1} X'$  Bousfield shows that  $\pi_i(F_b, b) \cong N^k \pi_{i+k}(X', b)$  [B3, 10.2]. Here,  $N^k \pi_{i+k}(X', b) \cong \pi_{i+k}(X^k, b_k) \cap \ker s^0 \cap \cdots \cap \ker s^{k-1}$ . Thus the normalization is finite for each  $i \geq 0$ . Hence  $\pi_i(F_b, b)$  is finite for  $i \geq 0$  and for each choice of  $b$ . Because  $\text{Tot}_{k-1} X'$  has finite homotopy this is enough to show that  $\pi_i(\text{Tot}_k X', b)$  is finite for each  $i \geq 0$  and for each choice of  $b$ .  $\square$

**3. The Eilenberg-Moore spectral sequence.** In this section, we generalize the Eilenberg-Moore spectral sequence results stated by Bousfield [B2, 4.1, 8.4]. Theorem 3.3 is used in section 4 to prove Theorem 4.1 and in section 5 to prove Theorem 5.3.

Consider the fibre square

$$\begin{array}{ccc} M & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X & \longrightarrow & B, \end{array}$$

where  $f$  is a fibration and  $M$  is the pull-back. Farjoun and Smith noticed that for  $X$  contractible they could generalize W. Dwyer's Eilenberg-Moore spectral sequence convergence result to cases where  $B$  is not necessarily connected [D1], [FS]. Bousfield considered the convergence of the Eilenberg-Moore spectral sequence for a fibre square with  $X$ ,  $Y$ , and  $B$  connected [B2, 4.1]. We combine these two directions in the next theorem. Let  $Y_y$  denote the component of  $Y$  containing the point  $y$ . Let  $F_y$  be the fibre of the fibration  $Y_y \rightarrow B$  over the point  $f(y)$ .

**THEOREM 3.1.** *In the above diagram assume that  $\pi_0 X \times_{\pi_0 B} \pi_0 Y$  is finite and that  $\pi_1(B, f(y))$  acts nilpotently on  $H_*(F_y)$  for every point  $y \in Y$  where  $Y_y$  is in*

the image of  $\pi_0 X \times_{\pi_0 B} \pi_0 Y$ . Then the Eilenberg-Moore spectral sequence for this diagram strongly converges to  $H_* M$ .

*Proof.* We will denote the cobar construction of the above diagram by  $B'$ . We can assume  $X$ ,  $Y$ , and  $B$  are fibrant simplicial sets. This ensures that  $B'$  is fibrant. Let  $X = \coprod_{\alpha} X_{\alpha}$  where  $\{X_{\alpha}\}$  is the set of connected components of  $X$ . Similarly, let  $Y = \coprod_{\beta} Y_{\beta}$ . Then it is easy to see that  $B' = \coprod_{\alpha, \beta} B'_{\alpha, \beta}$  where  $B'_{\alpha, \beta}$  is the cobar construction of the following diagram.

$$(1) \quad \begin{array}{ccc} & & Y_{\beta} \\ & & \downarrow \\ X_{\alpha} & \longrightarrow & B \end{array}$$

We show that the spectral sequence for each  $B'_{\alpha, \beta}$  strongly converges. Then we use this information to conclude the statement of the theorem.

We consider two different cases for the cosimplicial space  $B'_{\alpha, \beta}$ . The first case to consider is when the images of  $X_{\alpha}$  and  $Y_{\beta}$  lie in different components of  $B$ . In this case the pull-back is empty. Thus  $\text{Tot } B'_{\alpha, \beta} = \emptyset$ . A computation shows that the  $E^2$ -term of the spectral sequence is 0. Hence the spectral sequence is strongly convergent.

Next consider the case when the images of  $X_{\alpha}$  and  $Y_{\beta}$  lie in the same component of  $B$ . Call this component  $B_0$ . Let  $B'_0$  be the cobar construction for the following diagram.

$$(2) \quad \begin{array}{ccc} & & Y_{\beta} \\ & & \downarrow f_{\beta} \\ X_{\alpha} & \longrightarrow & B_0 \end{array}$$

The inclusion  $B_0 \rightarrow B$  induces an inclusion of cosimplicial spaces  $i : B'_0 \rightarrow B'_{\alpha, \beta}$ .

By assumption,  $\pi_1 B_0$  acts nilpotently on  $H_*(f_{\beta}^{-1} *)$ . Thus, by [B2, 4.1], this spectral sequence strongly converges for  $B'_0$ . This is equivalent to the top map in the following diagram being a pro-isomorphism for each  $n$ .

$$\begin{array}{ccc} \{H_n \text{Tot } B'_0\} & \longrightarrow & \{H_n T_s(R \otimes B'_0)\} \\ \downarrow & & \downarrow \\ \{H_n \text{Tot } B'_{\alpha, \beta}\} & \longrightarrow & \{H_n T_s(R \otimes B'_{\alpha, \beta})\} \end{array}$$

By considering the contribution of the homology of the components of  $B$  other than  $B_0$  we see that the inclusion  $i$  induces an  $E^2$ -isomorphism for each  $n$ . This implies that the right map above is a pro-isomorphism by Lemma 2.2. The homotopy pull-back for diagram (1) is homotopy equivalent to the homotopy pull-back for diagram (2). Thus  $\text{Tot } B'_0 \rightarrow \text{Tot } B'_{\alpha,\beta}$  is a homotopy equivalence. This shows that the left map above is a pro-isomorphism. Thus we conclude that the bottom map is a pro-isomorphism. This shows that the spectral sequence for  $B'_{\alpha,\beta}$  strongly converges.

To conclude the proof we need to show that since each  $B'_{\alpha,\beta}$  is strongly convergent,  $B'$  is also strongly convergent. Because  $\Delta'$  has connected codegrees  $\text{Tot}$  commutes with coproducts. So  $\text{Tot } B' = \coprod_{\alpha,\beta} \text{Tot } B'_{\alpha,\beta}$  because  $B' = \coprod_{\alpha,\beta} B'_{\alpha,\beta}$ . This also shows that the  $E^2$ -term for the cosimplicial space  $B'$  splits as the direct sum of the  $E^2$ -terms for each  $B'_{\alpha,\beta}$ . Hence we have the following commutative diagram.

$$\begin{array}{ccc}
 \bigoplus_{\alpha,\beta} \{H_* \text{Tot } B'_{\alpha,\beta}\} & \longrightarrow & \bigoplus_{\alpha,\beta} \{H_* T_s(R \otimes B'_{\alpha,\beta})\} \\
 \downarrow & & \downarrow \\
 \{H_* \text{Tot } B'\} & \longrightarrow & \{H_* T_s(R \otimes B')\}
 \end{array}$$

The top map is a pro-isomorphism on each direct summand because each  $B'_{\alpha,\beta}$  strongly converges. Since  $\pi_0 X \times_{\pi_0 B} \pi_0 Y$  is finite and the spectral sequence collapses at  $E^2$  when  $\text{Tot } B'_{\alpha,\beta}$  is empty, the top map is in fact a pro-isomorphism. Arguments similar to those above show that both the left and right maps are pro-isomorphisms. Thus the bottom map is a pro-isomorphism. This is equivalent to the statement of the theorem.  $\square$

Using this convergence result for the Eilenberg-Moore spectral sequence for spaces we can prove the following theorem for cosimplicial spaces. This is a generalization of [B2, 8.4].

**THEOREM 3.2.** *Let*

$$\begin{array}{ccc}
 M' & \longrightarrow & Y' \\
 \downarrow & & \downarrow f \\
 X' & \longrightarrow & B'
 \end{array}$$

*be a fibre square of cosimplicial spaces where  $f$  is a fibration,  $M'$  is the pull-back, and  $X'$ ,  $Y'$ , and  $B'$  are fibrant and pro-convergent. Assume that  $\pi_0 X^n \times_{\pi_0 B^n} \pi_0 Y^n$*

is finite,  $\pi_0 \text{Tot}_s X' \times_{\pi_0 \text{Tot}_s B'} \pi_0 \text{Tot}_s Y'$  is finite, and that  $\pi_1(B^n, *)$  and  $\pi_1(\text{Tot}_s B', *)$  are finite  $p$ -groups for each  $n, s$ , and choice of base points  $* \in B^n$  and  $* \in \text{Tot}_s B'$ . Then  $M'$  is pro-convergent.

*Proof.* Using the proof of [B2, 8.4], we see that  $M'$  is pro-convergent if  $X', Y'$ , and  $B'$  are pro-convergent and the Eilenberg-Moore spectral sequences for the following diagrams are strongly convergent

$$\begin{array}{ccc} & \text{Tot}_s Y' & \\ & \downarrow & \\ \text{Tot}_s X' & \longrightarrow & \text{Tot}_s B' \end{array} \qquad \begin{array}{ccc} & Y^n & \\ & \downarrow & \\ X^n & \longrightarrow & B^n \end{array}$$

for each  $s$  and  $n$ . A finite  $p$ -group acts nilpotently on any  $\mathbb{F}_p$  vector space. Hence a finite  $p$ -group always acts nilpotently on mod  $p$  homology. Thus, by Theorem 3.1, we see that the hypotheses of this theorem ensure that these spectral sequences are in fact strongly convergent.  $\square$

We need to apply this convergence result to a pull-back diagram where the cosimplicial spaces are not necessarily fibrant and the maps are not necessarily fibrations. The following theorem is a slight generalization of Theorem 3.2 which is tailored to this application.

THEOREM 3.3. *Let*

$$\begin{array}{ccc} M' & \longrightarrow & Y' \\ \downarrow & & \downarrow f \\ X' & \longrightarrow & B' \end{array}$$

be a pull-back diagram of cosimplicial spaces where  $f^n : Y^n \longrightarrow B^n$  is a fibration for each  $n$ ,  $B'$  is fibrant, and  $X', Y'$ , and  $B'$  are pro-convergent. Assume that  $\pi_0 \bar{X}^n \times_{\pi_0 B^n} \pi_0 \bar{Y}^n$  is finite,  $\pi_0 \text{Tot}_s \bar{X}' \times_{\pi_0 \text{Tot}_s B'} \pi_0 \text{Tot}_s \bar{Y}'$  is finite, and  $\pi_1(B^n, *)$  and  $\pi_1(\text{Tot}_s B', *)$  are finite  $p$ -groups for each  $n, s$  and choice of base points  $* \in B^n$  and  $* \in \text{Tot}_s B'$ . Then  $M'$  is pro-convergent.

*Proof.* We use the model category structure on cosimplicial spaces to replace the given diagram by a diagram which satisfies the hypotheses of Theorem 3.2 Let  $X' \xrightarrow{\cong} \bar{X}' \twoheadrightarrow B'$  be the factorization of  $X' \rightarrow B'$  into a trivial cofibration followed by a fibration. Repeat this process for  $Y' \rightarrow B'$ .

Consider the following diagram of pull-back squares

$$\begin{array}{ccccc}
 M' & \longrightarrow & W' & \longrightarrow & Y' \\
 \downarrow & & \downarrow & & \downarrow \simeq \\
 V' & \longrightarrow & Z' & \longrightarrow & \bar{Y}' \\
 \downarrow & & \downarrow & & \downarrow \\
 X' & \xrightarrow{\simeq} & \bar{X}' & \longrightarrow & B'.
 \end{array}$$

Because  $B'$  is fibrant,  $\bar{X}'$  and  $\bar{Y}'$  are fibrant. Thus the lower right hand pull-back diagram satisfies the hypotheses of Theorem 3.2. This implies that  $Z'$  is pro-convergent.

To conclude the proof of this lemma we only need to see that  $Z'$  is a weakly equivalent fibrant object for  $M'$ . Since a pull-back of a fibration is a fibration,  $Z' \longrightarrow \bar{Y}'$  is a fibration. So  $Z'$  is fibrant.

The model category on simplicial sets is *proper*, i.e., the pull-back of a weak equivalence along a fibration is a weak equivalence (and the dual statement). Thus since a fibration of fibrant cosimplicial spaces induces level-wise fibrations and weak equivalences are defined as level-wise weak equivalences,  $W' \xrightarrow{\simeq} Z'$  is a weak equivalence.

Now we need to see that  $M' \longrightarrow W'$  is a weak equivalence. By the hypotheses,  $Y^n \longrightarrow B^n$  is a fibration for each  $n$ . Thus  $W^n \longrightarrow \bar{X}^n$  is a fibration for each  $n$ . Each  $X^n \xrightarrow{\simeq} \bar{X}^n$  is a weak equivalence. Hence  $M^n \xrightarrow{\simeq} W^n$  is a weak equivalence because it is the pull-back of a weak equivalence along a fibration.  $\square$

**4. Cosimplicial simplicial abelian groups.** This section is devoted to proving Theorem 4.1 which shows that certain cosimplicial simplicial abelian groups are pro-convergent.

**THEOREM 4.1.** *Let  $B'$  be in  $csA$ . Assume that  $N^m \pi_n(B')$  is a finite  $p$ -group for each  $n \leq m$ . Then  $B'$  is pro-convergent as a cosimplicial space.*

We need the following lemmas before beginning the proof of Theorem 4.1.

**LEMMA 4.2.** *Let  $B'$  be in  $csA$ .  $\pi_0 B^m$  is a finite  $p$ -group for each  $m \geq 0$  if and only if  $N^m \pi_0 B'$  is a finite  $p$ -group for each  $m \geq 0$ .*

*Proof.* The forward direction is obvious. The converse follows by induction using the fibration  $N^m B' \rightarrow B^m \rightarrow M^{m-1}(B')$  and the fact that  $M^m \pi_i = \pi_i M^m$  [BK, X 6.3].  $\square$

In the next lemma we use the classifying space functor  $\overline{W}$ .  $\overline{W}$  and the associated total space functor  $W$  are defined on  $\text{sa}$  in [M]. These functors prolong to functors on  $\text{csa}$  [Do]. Let  $\Sigma$  be the functor which shifts grading by one. The following lemma states one useful fact about  $\overline{WB}$ .

LEMMA 4.3 For  $B'$  in  $\text{csa}$ ,  $H_i T_m \overline{WB}' = H_{i-1} T_m B'$ .

*Proof.* For  $G$  in  $\text{sa}$ , the definitions of  $\overline{W}$  and  $N_*$  imply that  $N_* \overline{WG} = \Sigma N_* G$ . Thus, for  $B'$  in  $\text{csa}$ ,  $N^* N_* \overline{WB}' = N^* \Sigma N_* B'$ . Hence,  $T_m \overline{WB}' = \Sigma T_m B'$ . The lemma follows by applying homology.  $\square$

To begin the proof of Theorem 4.1, we prove the following special case.

LEMMA 4.4. Let  $B'$  be in  $\text{csa}$  with  $N^m \pi_n B'$  a finite  $p$ -group for  $n \leq m$ . If there exists an  $N$  such that  $\{H_i T_m B'\}$  is pro-trivial for  $i < N$ , then  $B'$  is pro-convergent.

*Proof.* We use descending induction on  $N$ , beginning with  $N = 0$ . Given  $B'$  satisfying the hypotheses of the lemma for  $N = 0$ , consider  $\overline{WB}'$ . By Lemma 4.3 we see that  $\{H_i T_m \overline{WB}'\}$  is pro-trivial for  $i \leq 0$ . By construction  $\overline{WB}'$  is termwise connected, so  $\overline{WB}'$  is pro-convergent by the following lemma due to Bousfield.

LEMMA 4.5. [B2, 8.7] Let  $A'$  be in  $\text{csa}$ . If  $\{H_i T_m A'\}$  is pro-trivial for each  $i \leq 0$  and  $A'$  is termwise connected, then  $A'$  is pro-convergent.

Consider the following fibre square of cosimplicial spaces

$$\begin{array}{ccc} B' & \longrightarrow & WB' \\ \downarrow & & \downarrow \\ * & \longrightarrow & \overline{WB}' \end{array}$$

where  $*$  denotes the constant cosimplicial point. We need to verify that the hypotheses for applying Theorem 3.2 to this fibre square are satisfied.  $WB'$  is pro-convergent because it is weakly equivalent to the cosimplicial point. This also implies that  $\pi_0 WB'^n$  and  $\pi_0 \text{Tot}_s WB'$  are trivial. Hence the fibred product of components for both codegrees and the partial total spaces of this fibre square are trivial.

From the proof of Lemma 4.3 we see that  $N_* \overline{WB}' = \Sigma N_* B'$ . Hence  $\pi_* \overline{WB}' = \Sigma \pi_* B'$ . So  $N^m \pi_n \overline{WB}' = N^m \pi_{n-1} B'$  is a finite  $p$ -group for each  $n \leq m + 1$ . This is enough to ensure that  $\pi_1 \text{Tot}_m \overline{WB}'$  is a finite  $p$ -group for  $m \geq 0$  [BK, X 6.3]. Since  $N^m \pi_0 B'$  is a finite  $p$ -group for  $m \geq 0$ , Lemma 4.2 implies that  $\pi_0 B'^m$  is a finite  $p$ -group for  $m \geq 0$ . Thus, since  $\pi_1 \overline{WB}'^m = \pi_0 B'^m$ ,  $\pi_1 \overline{WB}'^m$  is a finite  $p$ -group for each  $m \geq 0$ . Since  $WB'$  and  $\overline{WB}'$  are in  $\text{csa}$  and  $WB' \rightarrow \overline{WB}'$  is an epimorphism,

$WB'$  and  $\overline{WB}'$  are fibrant and  $WB' \rightarrow \overline{WB}'$  is a fibration [BK, X 4.9]. Thus Theorem 3.2 applies to the fibre square above. Hence  $B'$  is pro-convergent.

For the induction step, assume that we have proved Lemma 4.4 for  $n > N$ . Given  $B'$  satisfying the hypotheses of the lemma for  $n = N$ , we see by Lemma 4.3 that  $\overline{WB}'$  satisfies the hypotheses for  $n = N+1$ . Thus, by the induction assumption,  $\overline{WB}'$  is pro-convergent. As in the case for  $N = 0$ , the fibre square satisfies the conditions for Theorem 3.2. So we conclude that  $B'$  is pro-convergent.  $\square$

*Proof of Theorem 4.1.* To finish the proof of Theorem 4.1 we need to consider a map of towers of cosimplicial spaces and apply Corollary 2.5.

Let  $X'$  be a cosimplicial space. The  $n$ th cosimplicial skeleton,  $\text{cosk}_n(X')$ , is the cosimplicial space generated by all simplices of  $X'$  of codegree less than or equal to  $n$ . This is just the  $n$ th skeleton of  $X'$  considered as a simplicial object over the opposite category of spaces. Note that  $X^m \rightarrow (\text{cosk}_n X')^m$  is an isomorphism for  $n \geq m$ .

Consider the map of towers of cosimplicial spaces  $\{B'\} \rightarrow \{\text{cosk}_n B'\}$  for  $B'$  satisfying the conditions of Theorem 4.1. This is a pro-homotopy equivalence on each codegree. To use Corollary 2.5, we need to see that each  $\text{cosk}_n B'$  is pro-convergent.  $N^j \text{cosk}_n B'$  is trivial for  $j > n$ . So  $H_* N^j \text{cosk}_n B'$  is trivial for  $j > n$ . Because the cosimplicial and simplicial operators commute  $H_* N^j = N^j H_*$ . Thus  $E_{j,*}^1(\text{cosk}_n B') = 0$  for  $j > n$ . Hence  $E_{s,s+i}^1(\text{cosk}_n B') = 0$  for  $i < -n$ , since  $E_{p,q}^1 = 0$  if  $p$  or  $q$  is negative. By Lemma 2.3 this shows that  $\{H_i T_m \text{cosk}_n B'\}$  is pro-trivial for  $i < -n$ .

$N^i \pi_k \text{cosk}_n B'$  is trivial for  $i > n$  and is equal to  $N^i \pi_k B'$  for  $i \leq n$ . So, for any  $k \leq i$ ,  $N^i \pi_k \text{cosk}_n B'$  is a finite  $p$ -group. Hence, by Lemma 4.4, we see that each  $\text{cosk}_n B'$  is pro-convergent. Thus, by Corollary 2.5,  $B'$  is pro-convergent.  $\square$

*Remark.* We should note here that in fact the hypotheses of Theorem 4.1 can be weakened. As is evident from the statement of Proposition 2.4, it is only necessary for  $\text{cosk}_n B'$  to be pro-convergent for infinitely many  $n$ . If  $N^m \pi_0 B'$  is a finite  $p$ -group for each  $m \leq n$  and  $\pi_k \text{Tot}_n \overline{W}^{n+1} B'$  is a finite  $p$ -group for  $1 \leq k \leq n$ , then  $\text{cosk}_n B'$  is pro-convergent. This implies, for instance, that if  $N^m \pi_n B'$  is a finite  $p$ -group except for finitely many  $m$  and  $n$  and there exists some  $r$  such that each  $E_{s,t}^r(B')$  is a finite  $p$ -group in the homotopy spectral sequence of  $B'$ , then  $B'$  is pro-convergent.

**5. Pro-convergence.** In this section we use Theorem 4.1 to prove Theorem 5.3, a pro-convergence statement which applies to a more general class of cosimplicial spaces.

A connected space  $X$  is *nilpotent* if its fundamental group acts nilpotently on each  $\pi_i X$  for  $i \geq 1$ . A connected space  $X$  is *p-nilpotent* if it is nilpotent and

$\pi_i X$  is a  $p$ -group with bounded torsion for each  $i$ . In general, define a space to be  $p$ -nilpotent if each of its components is  $p$ -nilpotent. See also [BK, III 5].

We need a construction from [BK, I 2]. Let  $R \otimes X$  be the simplicial vector space generated by the simplicial set  $X$ . Define  $RX \subset R \otimes X$  to be the simplicial set consisting of the simplices  $\Sigma r_i x_i$  with  $\Sigma r_i = 1$ . Then  $RX$  has an affine  $R$ -structure which becomes an  $R$ -module structure once a base point is chosen in  $X$ . See [BK, I 2.2].  $R$  is a triple on the category of spaces. So iterating  $R$  produces an augmented cosimplicial space,  $X \rightarrow R'X$ , called the  $p$ -resolution of  $X$ . Here  $(R'X)^n = R^{n+1}X$ . Let  $R_s X = \text{Tot}_s R'X$ .

Bousfield and Kan show that a connected space is  $p$ -nilpotent if and only if  $\{X\} \rightarrow \{R_s X\}$  is a weak pro-homotopy equivalence [BK, III 5.3]. To prove Theorem 5.3 below we need to prove the same statement for spaces with finitely many components. Before we can do this we need the following lemma about the interaction of the  $p$ -resolution and coproducts. We also include in this lemma a similar statement about products which will be used in section 8.

LEMMA 5.1. *The product of the projection maps induces a weak pro-homotopy equivalence  $\{R_s(X \times Y)\} \rightarrow \{R_s X \times R_s Y\}$ . Consider  $Z = \coprod Z_a$ , where  $Z_a$  is the component corresponding to  $a \in \pi_0 Z$ . The map of towers induced by inclusion  $\{\coprod R_s Z_a\} \rightarrow \{R_s Z\}$  is a weak pro-homotopy equivalence.*

*Proof.* In [BK, I 7.1 and 7.2] Bousfield and Kan show that the  $p$ -completion functor commutes with products and coproducts. This lemma is just the tower theoretic analogue to those statements. In fact, for showing that  $p$ -completion commutes with products they show that there is an  $E^2$ -isomorphism of homotopy spectral sequences between  $R'(X \times Y)$  and  $R'X \times R'Y$ . As in Lemma 2.2, translating this  $E^2$ -isomorphism into tower theoretic language shows that  $\{\pi_i \text{Tot}_s R'(X \times Y)\} \rightarrow \{\pi_i \text{Tot}_s(R'X \times R'Y)\}$  is a pro-isomorphism for each  $i$ . This proves the first statement in the lemma.

Similarly, for coproducts Bousfield and Kan exhibit an  $E^2$ -isomorphism which translates into tower theoretic language to show that  $\{\text{Tot}_s(\coprod R'Z_a)\} \rightarrow \{R_s Z\}$  is a weak pro-homotopy equivalence. Since  $\text{Tot}_0 X' = X^0$ , the functor  $\text{Tot}_0$  commutes with coproducts. For  $s > 0$ , the simplicial  $s$ -skeleton of  $\Delta'$  has connected codegrees. So  $\text{Tot}_s$  also commutes with coproducts. Thus  $\{\coprod R_s Z_a\} \rightarrow \{R_s Z\}$  is a weak pro-homotopy equivalence.  $\square$

Now we are ready to prove the next lemma.

LEMMA 5.2. *If  $\{X\} \rightarrow \{R_s X\}$  is a weak pro-homotopy equivalence then  $X$  is  $p$ -nilpotent. A space with finitely many components is  $p$ -nilpotent if and only if  $\{X\} \rightarrow \{R_s X\}$  is a weak pro-homotopy equivalence.*

*Proof.* Write  $X$  as a coproduct of its components, i.e.,  $X = \coprod X_a$  with each  $X_a$  a connected component. To prove each of the statements in the lemma we

consider the following diagram.

$$\begin{array}{ccc}
 \{\coprod X_a\} & \longrightarrow & \{X\} \\
 \downarrow & & \downarrow \\
 \{\coprod R_s X_a\} & \longrightarrow & \{R_s X\}
 \end{array}$$

Lemma 5.1 shows that the bottom map in this diagram is a weak pro-homotopy equivalence. Assume that  $\{X\} \rightarrow \{R_s X\}$  is a weak pro-homotopy equivalence. Using the diagram above, this implies that  $\{\coprod X_a\} \rightarrow \{\coprod R_s X_a\}$  is a weak pro-homotopy equivalence. Because this map is a coproduct of maps each component map is also a weak pro-homotopy equivalence. Thus each  $X_a$  is  $p$ -nilpotent by [BK, III 5.3]. By definition this shows that  $X$  is  $p$ -nilpotent.

Assume  $X$  is  $p$ -nilpotent, then by definition each  $X_a$  is  $p$ -nilpotent. This, by [BK III 5.3], implies that each  $\{X_a\} \rightarrow \{R_s X_a\}$  is a weak pro-homotopy equivalence. The coproduct of finitely many weak pro-homotopy equivalences is again a weak pro-homotopy equivalence. Hence if  $X$  has finitely many components  $\{X\} \rightarrow \{R_s X\}$  is a weak pro-homotopy equivalence because the other three maps in the diagram are also weak pro-homotopy equivalences.  $\square$

The next statement is the main pro-convergence theorem.

**THEOREM 5.3.** *Let  $X'$  be a cosimplicial space. Assume  $X^n$  is  $p$ -nilpotent and  $H_* X^n$  is finite type for all  $n$ . Then  $X'$  is pro-convergent. In other words, the following map is a pro-isomorphism*

$$\Phi : \{H_* \text{Tot}_s \bar{X}'\} \rightarrow \{H_* T_s(R \otimes \bar{X}')\}.$$

The following lemma contains the main work for proving Theorem 5.3.

First consider the bicosimplicial space  $R'X'$  with codegrees  $(R'X')^{s,n} = R^{s+1}X^n$ . We refer to the cosimplicial direction within  $X'$  as the vertical direction and the other cosimplicial direction as the horizontal direction. Then by definition  $R_s X'$ , with codegrees  $(R_s X')^n = R_s X^n$ , is  $\text{Tot}_s^h R'X'$ .

**LEMMA 5.4.** *If  $X'$  is a pointed cosimplicial space such that  $H_* X^n$  is finite type, then  $R_s X'$  is pro-convergent for all  $s < \infty$ .*

*Proof.* We prove this lemma by induction on  $s$ . First consider  $s = 0$ .  $R_0 X'$  is  $R^1 X'$ . Given a pointed cosimplicial space  $X'$ ,  $R^1 X'$  is in  $\text{csA}$  [BK, I 2.2].

Since  $\pi_m R^1 X^n = \overline{H}_m X^n$ , this group is a finite  $p$ -group by hypothesis. Thus, by Theorem 4.1,  $R^1 X'$  is pro-convergent.

Now we assume by induction that  $R_s X'$  is pro-convergent and prove that  $R_{s+1} X'$  is pro-convergent. The following diagram forms  $R_{s+1} X'$  as a pull-back [B1, pp. 149–150].

$$(3) \quad \begin{array}{ccc} R_{s+1} X' \simeq \text{Tot}_{s+1}^h R' X' & \longrightarrow & \text{Hom}(\Delta[s+1], R^{s+2} X') \\ \downarrow & & \downarrow \\ R_s X' \simeq \text{Tot}_s^h R' X' & \longrightarrow & P' \end{array}$$

Here  $P'$  is the pull-back of the following diagram

$$(4) \quad \begin{array}{ccc} & & \text{Hom}(\Delta[s+1], M^s(R' X')) \\ & & \downarrow \\ \text{Hom}(\partial\Delta[s+1], R^{s+2} X') & \longrightarrow & \text{Hom}(\partial\Delta[s+1], M^s(R' X')) \end{array}$$

where  $\partial\Delta[s]$  is the  $s - 1$  skeleton of  $\Delta[s]$ . We use Theorem 3.3 to prove that  $R_{s+1} X'$  is pro-convergent. Hence, we need to see that the other three cosimplicial spaces in diagram (3) are pro-convergent,  $P'$  is fibrant, the right map is a level-wise fibration, certain spaces have finitely many components, and that certain fundamental groups are finite  $p$ -groups.

First, we assume that  $R_s X'$  is pro-convergent by way of induction hypothesis. Note that  $R^s X'$  is in cSA for any  $s$ . Using the fact that a space with finite homotopy groups has finite homology groups, we see that each  $\pi_m R^s X^n$  is a finite  $p$ -group. Thus Theorem 4.1 shows that  $R^s X'$  is pro-convergent for any  $s$ .  $\text{Hom}(\Delta[s+1], R^{s+2} X')$  also satisfies the hypotheses for Theorem 4.1. Hence it is pro-convergent.

Applying Lemma 2.6 to  $R' X^n$  we see that  $\pi_m R_s X^n$  is finite for each  $m, s, n$  and choice of base point. This is also true of the fibrant replacement,  $\overline{R_s X'}$ . Thus applying Lemma 2.6 to  $\overline{R_s X'}$  shows that  $\pi_m \text{Tot}_t(\overline{R_s X'})$  is finite for each  $m, t, s$  and choice of base point. Since  $\text{Hom}(\Delta[s+1], R^{s+2} X')$  is fibrant, applying Lemma 2.6 shows that  $\pi_m \text{Tot}_t \text{Hom}(\Delta[s+1], R^{s+2} X')$  is finite for each  $m, t, s$  and choice of base point. In particular, these arguments show that  $\pi_0 \text{Tot}_t \overline{R_s X'} \times_{\pi_0 \text{Tot}_t P'} \pi_0 \text{Tot}_t \text{Hom}(\Delta[s+1], R^{s+2} X')$  and  $\pi_0 \overline{R_s X^n} \times_{\pi_0 P^n} \pi_0 \text{Hom}(\Delta[s+1], R^{s+2} X^n)$  are finite for each  $s, t$ , and  $n$ .

Next we show that  $\text{Hom}(\Delta[s+1], R^{s+2} X^n) \rightarrow P^n$  is a fibration for each  $n$ . The maps  $s : R^{s+2} X^n \rightarrow M^s(R' X^n)$  and  $i : \partial\Delta[s+1] \rightarrow \Delta[s+1]$  induce map  $(i, s) : \text{Hom}(\Delta[s+1], R^{s+2} X^n) \rightarrow P^n$ . The model category on cosimplicial

spaces developed in [BK, X] is a simplicial model category. Hence the axiom for a simplicial model category shows that if  $s$  is a fibration and  $i$  is a cofibration then  $\text{map}(i, s)$  is a fibration [BK, X 5]. Since  $R'X^n$  is fibrant [BK, X 4.10]  $s : R^{s+2}X^n \rightarrow M^s(R'X^n)$  is a fibration. The map  $i : \partial\Delta[s+1] \rightarrow \Delta[s+1]$  is a cofibration. Thus  $\text{Hom}(\Delta[s+1], R^{s+2}X^n) \rightarrow P^n$  is a fibration.

We are left with proving that  $P'$  is pro-convergent and  $\pi_1 P^n$  and  $\pi_1 \text{Tot}_s P'$  are finite  $p$ -groups for each  $n$  and  $s$ . The horizontal codegeneracy maps in  $R'X'$  are homomorphisms. Thus  $M^s(R'X')$  is in  $\text{csA}$ . Hence diagram (4) is a pull-back square in  $\text{csA}$ . So  $P'$  is in  $\text{csA}$ . The finite type hypothesis ensures that the necessary homotopy groups are finite  $p$ -groups. Hence, by Theorem 4.1,  $P'$  is pro-convergent.

Using pull-back diagrams to inductively build  $\text{Tot}_s P'$  we see for each  $s$  that the fundamental group of each component of  $\text{Tot}_s P'$  is a finite  $p$ -group. So  $\pi_1(P^n, *)$  and  $\pi_1(\text{Tot}_s P', *)$  are finite  $p$ -groups for each  $n, s$  and choice of base points. This is what we needed to apply Theorem 3.3 to diagram (3) and conclude that  $R_{s+1}X'$  is pro-convergent.  $\square$

*Proof of Theorem 5.3.* To use Lemma 5.4 we need a pointed cosimplicial space. If  $X'$  is not pointed then consider  $Y' = X' \coprod *$ , where  $*$  is the constant cosimplicial space with each codegree equal to a point. If  $X'$  satisfies the hypotheses in Theorem 5.3 then  $Y'$  does too. Also note that  $\bar{Y}' = \bar{X}' \coprod *$ . Because  $R \otimes \bar{Y}' = (R \otimes \bar{X}') \oplus (R \otimes *)$  and  $\text{Tot}_s \bar{Y}' = \text{Tot}_s \bar{X}' \coprod *$ , the vertical maps in the diagram below are level-wise isomorphisms.

$$\begin{array}{ccc} \{H_* \text{Tot}_s \bar{Y}'\} & \longrightarrow & \{H_* T_n(R \otimes \bar{Y}')\} \\ \downarrow & & \downarrow \\ \{H_* \text{Tot}_s \bar{X}' \oplus H_* *\} & \longrightarrow & \{H_* T_n(R \otimes \bar{X}') \oplus H_* T_n(R \otimes *)\} \end{array}$$

Because  $\{H_*(*)\} \rightarrow \{H_* T_n(R \otimes *)\}$  is an isomorphism of constant towers this shows that  $X'$  is pro-convergent if and only if  $Y'$  is pro-convergent. Thus we can assume that  $X'$  is a pointed cosimplicial space.

To finish the proof of Theorem 5.3 we use Corollary 2.5. Consider the map of towers of pointed cosimplicial spaces  $\{X'\} \rightarrow \{R_s X'\}$ . Lemma 5.4 shows that for a pointed cosimplicial space  $X'$  satisfying the hypotheses in Theorem 5.3 each  $R_s X'$  is pro-convergent. Since each  $X^m$  is  $p$ -nilpotent and has finitely many components, Lemma 5.2 shows that  $\{X^m\} \rightarrow \{R_s X^m\}$  is a weak pro-homotopy equivalence. Thus, Corollary 2.5 shows that  $X'$  is pro-convergent.  $\square$

*Remarks.* Given the hypotheses of Theorem 4.1, one might expect that Theorem 5.3 would only require  $H_m X^n$  finitely generated for  $m \leq n$ . In fact, here

we need that  $H_*X^n$  is finite type, because the construction of the matching space  $M^sX$  uses codegrees  $X^n$  for  $n \leq s$ . See [BK, X 6.3].

We should also note here that the finite type assumptions are necessary. Consider for example the Eilenberg-Moore spectral sequence for the path loop fibration over the classifying space of an infinite dimensional  $\mathbb{Z}/2$  vector space  $V$ .

$$\begin{array}{ccc} \Omega BV & \longrightarrow & PBV \\ & & \downarrow \\ & & BV \end{array}$$

Let  $B'$  be the cobar construction for this fibration. At  $E^\infty$ , this spectral sequence has infinitely many nonzero filtrations on the zero total degree line. Thus, by Lemma 2.3,  $\{H_*T_s(R \otimes B')\}$  is not pro-constant. But  $\{H_*(\text{Tot}_s B')\}$  for any cobar construction is pro-constant since  $\text{Tot}_1 B' \simeq \text{Tot}_2 B' \simeq \dots \simeq \text{Tot} B'$ . Thus  $\Phi : \{H_*(\text{Tot}_s B')\} \rightarrow \{H_*T_s(R \otimes B')\}$  is not a pro-isomorphism.

**6. Strong convergence.** In this section we discuss strong convergence of the homology spectral sequence for a cosimplicial space. As an example we then apply this convergence result to the calculation of the homology of a mapping space.

First we need two definitions. In section 5 we defined the cosimplicial space  $R'X$ . Its total space,  $\text{Tot} R'X = R_\infty X$ , is called the *p-completion* of the space  $X$ . A space is called *p-good* if the map from the space to its p-completion is a homology isomorphism, i.e.  $H_*X \xrightarrow{\cong} H_*R_\infty X$ .

We combine Theorem 5.3 and Corollary 1.2 from [S] to get the following strong convergence result. To use Corollary 1.2 from [S], each  $\{H_i \text{Tot}_s \bar{X}'\}$  must be pro-isomorphic to a tower of finite groups. If  $H_*X^n$  is finite type then so is  $H_*\bar{X}^n$ . Thus  $\pi_m(R \otimes \bar{X}^n) = H_m\bar{X}^n$  is finite for each  $m, n$  and choice of base point. Hence Lemma 2.6 implies that  $\pi_m \text{Tot}_s(R \otimes \bar{X}')$  is finite for each  $m$  and  $s$ . Bousfield shows that  $\pi_m \text{Tot}_s(R \otimes \bar{X}') \cong H_m T_s(R \otimes \bar{X}')$  [B2, 2.2]. Thus  $\{H_i T_s(R \otimes \bar{X}')\}$  is a tower of finite groups for each  $i$ . So  $\{H_i \text{Tot}_s \bar{X}'\}$  is pro-isomorphic to a tower of finite groups.

**THEOREM 6.1.** *Let  $X'$  be a cosimplicial space with  $X^s$  p-nilpotent and  $H_*X^s$  finite type for each  $s$ . Assume either*

- (a)  $H_* \text{Tot} \bar{X}'$  is finite type, or
- (b)  $\varprojlim H_* \text{Tot}_s \bar{X}'$  is finite type.

*Then the homology spectral sequence for  $X'$  is strongly converging to  $H_* \text{Tot} \bar{X}'$  (i.e.  $\{H_* \text{Tot} \bar{X}'\} \rightarrow \{H_* T_s(R \otimes \bar{X}')\}$  is a pro-isomorphism) if and only if  $\text{Tot} \bar{X}'$  is p-good.*

*Remarks.* In section 10 we prove another strong convergence result which only requires that each  $X^s$  is  $p$ -complete.

We should note that since  $\{H_*\text{Tot}_s\overline{X}\}$  is pro-finite type—i.e., each degree is pro-isomorphic to a tower of finite groups—condition (b) is equivalent to  $\{H_*\text{Tot}_s\overline{X}\}$  being pro-constant. Thus, considering Lemma 2.3 and Theorem 5.3, we see that condition (b) is equivalent to having certain structural strong convergence properties. Specifically, condition (b) is equivalent to requiring that at each  $E_{s,t}$  there are only finitely many nonzero differentials and that “by  $E^\infty$ ” each total degree has only finitely many filtrations.

One common use for the homology spectral sequence of a cosimplicial space is calculating the homology of mapping spaces. We now consider the application of Theorem 6.1 to these calculations.

One cosimplicial space associated to a mapping space,  $\text{map}(X, Y)$ , is constructed by using the  $p$ -resolution of the target,  $\text{map}(X, R^s Y)$ . Note that each codegree here is  $p$ -nilpotent. If  $H_*X$  is finite and  $H_*Y$  is finite type, then  $\pi_* \text{map}(X, R^s Y)$  is finite type. Thus each codegree has finite type homology. Hence we have the following corollary to Theorem 6.1.

**COROLLARY 6.2.** *Let  $X$  and  $Y$  be spaces such that  $H_*X$  is finite and  $H_*Y$  is finite type. Assume either*

- (a)  $H_* \text{map}(X, R_\infty Y)$  is finite type, or
- (b)  $\varprojlim H_* \text{map}(X, R_s Y)$  is finite type.

*Then the homology spectral sequence for  $\text{map}(X, R^s Y)$  strongly converges to the  $H_* \text{map}(X, R_\infty Y)$  if and only if  $\text{map}(X, R_\infty Y)$  is  $p$ -good.*

*Remarks.* Again, note that condition (b) is equivalent to certain structural strong convergence conditions. Hence part (b) of this corollary states that if the spectral sequence “looks” like it is converging then it is strongly converging if and only if the mapping space is  $p$ -good. We should also note that this same result holds for pointed mapping spaces.

Under certain conditions one can easily see that  $\text{map}(X, R_\infty Y)$  is  $p$ -good. If  $Y$  is nilpotent then  $R_\infty Y$  is also nilpotent [BK, VI 5.1]. Hence, if  $X$  is a finite complex and  $Y$  is nilpotent,  $\text{map}(X, R_\infty Y)$  is nilpotent [BK, V 5.1] and therefore  $p$ -good [BK, VI 5.3].

**7. Exotic convergence.** In sections 5 and 6 we have analyzed when the homology spectral sequence is pro-convergent or strongly convergent. In this section, we change our focus. Here, instead of asking when the spectral sequence converges, we ask to what the spectral sequence is converging. This change in focus allows us to consider cosimplicial spaces whose codegrees are not necessarily  $p$ -nilpotent. The exotic convergence results are stated in Corollaries 7.6 and 7.7.

In section 5 we considered the bicosimplicial space  $R^\Delta X'$ . Here we consider the diagonal of this bicosimplicial space, which we denote  $R^\Delta X'$ . The  $p$ -resolution of the cosimplicial space  $X'$  is  $R^\Delta X'$ . The codegrees of this cosimplicial space are  $(R^\Delta X')^n = R^{n+1}X^n$ . Each codegree is a simplicial vector space over  $R$ . Therefore it is  $p$ -nilpotent. For a pointed cosimplicial space  $X'$ ,  $R^\Delta X'$  is group-like. Hence it is fibrant [BK, X 4.9]. In fact the following lemma shows that  $R^\Delta X'$  is fibrant for any cosimplicial space  $X'$ . So the  $p$ -resolution construction takes any cosimplicial space to a related fibrant cosimplicial space which has  $p$ -nilpotent codegrees.

LEMMA 7.1. *Let  $X'$  be a cosimplicial space.  $R^\Delta X'$  is fibrant.*

*Proof.* If  $X'$  is the empty cosimplicial space then  $R^\Delta X'$  is the cosimplicial point. Hence it is fibrant. Now assume  $X'$  is nonempty and hence also that  $X^0$  is nonempty. Choose an element  $x_0 \in X^0$ . This element is carried by the iterated coface operators  $d^i$  for  $i > 0$  to a system of basepoints. This system of basepoints is respected under all iterated cosimplicial operators not involving  $d^0$ . We will use this system of basepoints to show that  $R^\Delta X'$  is group-like and hence fibrant [BK, X 4.9].

Each  $R^{n+1}X^n$  has the structure of a simplicial  $R$ -module using the choice of basepoint above [BK, I 2.2]. To show that  $R^\Delta X'$  is group-like we need to see that each cosimplicial operator except  $d^0$  is a homomorphism with respect to these group structures. Each  $d^i : R^n X^{n-1} \rightarrow R^{n+1} X^n$  is a composite of  $d^i : R^n X^{n-1} \rightarrow R^n X^n$  and  $d^i : R^n X^n \rightarrow R^{n+1} X^n$ . Because each  $d^i : X^{n-1} \rightarrow X^n$  for  $i > 0$  preserves the chosen basepoints, each of these composite maps is a homomorphism. Similar arguments work for each  $s^i$  also. Thus any cosimplicial operator except  $d^0$  in  $R^\Delta X'$  is a homomorphism. Hence  $R^\Delta X'$  is fibrant.  $\square$

In order to apply Theorem 5.3 to the  $p$ -resolution we need the next lemma.

LEMMA 7.2. *Let  $X'$  be a cosimplicial space such that  $H_* X^n$  is finite type for each  $n$ . Then  $\pi_m (R^\Delta X')^n$  is finite and  $H_*(R^\Delta X')^n$  is finite type for each  $m$  and  $n$ .*

*Proof.* This is easy to see using induction, the fact that  $\overline{H}_* X^n = \pi_* R(X^n)$ , and the fact that a space with finite homotopy groups has finite mod  $p$  homology.  $\square$

The following corollary is an easy application of Theorem 5.3 given this lemma and the fact that each codegree of  $R^\Delta X'$  is  $p$ -nilpotent.

COROLLARY 7.3. *Let  $X'$  be a cosimplicial space. Assume that  $H_* X^n$  is finite type for all  $n$ . Then  $R^\Delta X'$  is pro-convergent. In other words, the following map is a pro-isomorphism*

$$\Phi : \{H_* \text{Tot}_s R^\Delta X'\} \rightarrow \{H_* T_s (R \otimes R^\Delta X')\}.$$

We can also apply Theorem 6.1 to the  $p$ -resolution to get the following corollary.

**COROLLARY 7.4.** *Let  $X'$  be a cosimplicial space with  $H_*X^n$  finite type for each  $n$ . Assume either*

- (a)  $H_*\text{Tot}R^\Delta X'$  is finite type, or
- (b)  $\varprojlim H_*\text{Tot}_s R^\Delta X'$  is finite type.

*Then  $\{H_*\text{Tot}R^\Delta X'\} \rightarrow \{H_*T_s(R \otimes R^\Delta X')\}$  is a pro-isomorphism, i.e., the homology spectral sequence for  $R^\Delta X'$  strongly converges, if and only if  $\text{Tot}R^\Delta X'$  is  $p$ -good.*

These corollaries become more interesting once we realize the relationship between a cosimplicial space and its  $p$ -resolution. Using the canonical map  $X \rightarrow R^s X$ , one can construct a map  $X' \rightarrow R^\Delta X'$ .

**LEMMA 7.5.** *Let  $X'$  be a cosimplicial space. The canonical map  $X' \rightarrow R^\Delta X'$  induces an isomorphism  $\pi^s H_t(X') \rightarrow \pi^s H_t(R^\Delta X')$  for all  $s$  and  $t$ .*

*Proof.* First consider the  $p$ -resolution of a space  $Y$ . Its homology is augmented by  $H_*Y \rightarrow H_*R'Y$ . The natural map  $R \otimes R'Y \rightarrow R \otimes Y$  induces a natural cosimplicial retraction of  $H_*R'Y$  onto  $H_*Y$ . Hence  $\pi^* H_*R'Y \cong H_*Y$ . This natural cosimplicial retraction can be extended to give a retraction of  $H_*R^\Delta X'$  onto  $H_*X'$ . Thus,  $\pi^* H_*X' \cong \pi^* H_*R^\Delta X'$ . See also [G, 3.4], where an analogous statement in cohomology is proved.  $\square$

This lemma shows that the map  $X' \rightarrow R^\Delta X'$  induces an  $E^2$ -isomorphism of the respective homology spectral sequences. Lemma 2.2 shows that the  $E^2$ -isomorphism induced by the map  $X' \rightarrow R^\Delta X'$  produces a pro-isomorphism  $\{H_*T_s(R \otimes X')\} \rightarrow \{H_*T_s(R \otimes R^\Delta X')\}$ . Thus strong convergence of the homology spectral sequence for  $R^\Delta X'$  is equivalent to having the homology spectral sequence for  $X'$  converge to  $H_*\text{Tot}R^\Delta X'$ . More precisely we have the following corollary.

**COROLLARY 7.6** *Under the hypotheses of Corollary 7.4, the following maps are both pro-isomorphisms*

$$\{H_*\text{Tot}R^\Delta X'\} \longrightarrow \{H_*T_s(R \otimes R^\Delta X')\} \longleftarrow \{H_*T_s(R \otimes X')\}$$

*if and only if  $\text{Tot}R^\Delta X'$  is  $p$ -good. In other words, the homology spectral sequence for  $X'$  is strongly converging to  $H_*\text{Tot}R^\Delta X'$  if and only if  $\text{Tot}R^\Delta X'$  is  $p$ -good.*

Restating Corollary 7.3 from this perspective, the following corollary states that the homology spectral sequence for  $X'$  pro-converges to the homology of the

tower  $\{\mathrm{Tot}_s R^\Delta X'\}$  quite generally.

**COROLLARY 7.7.** *Let  $X'$  be a cosimplicial space with  $H_*X^n$  finite type for each  $n$ . Then the following maps are both pro-isomorphisms.*

$$\{H_* \mathrm{Tot}_s R^\Delta X'\} \longrightarrow \{H_* T_s(R \otimes R^\Delta X')\} \longleftarrow \{H_* T_s(R \otimes X')\}$$

*Remark.* This corollary shows that for  $X'$  with  $H_*X^n$  finite type  $\{H_n T_s(R \otimes X')\}$  is pro-trivial for  $n < 0$ . Thus Lemma 2.3 shows that in this case the homology spectral sequence will vanish in negative total degrees “by  $E^\infty$ .” In [D3] W. Dwyer constructed operations which ensure over  $\mathbb{Z}/2$ , among other things, that  $E^\infty(X')$  vanishes in negative total degrees for any cosimplicial space  $X'$ . We conjecture that this is true over  $\mathbb{Z}/p$  for any prime  $p$ , although we have only proved this for cosimplicial spaces whose codegrees have finite type mod  $p$  homology.

**8. Exotic convergence of the Eilenberg-Moore spectral sequence.** In this section we apply the exotic convergence results discussed in section 7 to the Eilenberg-Moore spectral sequence. First we need the following proposition, which is useful for dealing with the total space of  $R^\Delta X'$ . Let  $Y''$  be a bicosimplicial space. Let  $\Delta Y''$  be the cosimplicial space with codegrees  $(\Delta Y'')^n = Y^{n,n}$ . Hence  $R^\Delta X' = \Delta R' X'$ . Let  $\mathrm{Tot}^h$  and  $\mathrm{Tot}^v$  refer to the “horizontal” and “vertical” Tot functors.

**PROPOSITION 8.1.**  *$\mathrm{Tot}(\Delta Y'')$  is isomorphic to  $\mathrm{Tot}^h \mathrm{Tot}^v(Y'')$  and  $\mathrm{Tot}^v \mathrm{Tot}^h(Y'')$ . Similarly,  $\{\mathrm{Tot}_s(\Delta Y'')\}$  is pro-isomorphic to  $\{\mathrm{Tot}_s^h \mathrm{Tot}_s^v Y''\}$  and  $\{\mathrm{Tot}_s^v \mathrm{Tot}_s^h Y''\}$ .*

*Proof.* Let  $\Delta' \times \Delta'$  be the bicosimplicial space such that  $(\Delta' \times \Delta')^{n,m} = \Delta[n] \times \Delta[m]$ . Then the first statement in the lemma is equivalent to  $\mathrm{Hom}(\Delta', \Delta Y'') \cong \mathrm{Hom}(\Delta' \times \Delta', Y'')$ . We exhibit the isomorphism on the zero simplices.

A  $k$ -simplex of  $(\Delta' \times \Delta')^{n,m}$  is a simplex  $\sigma_k \times \tau_k$  with  $\sigma_k \in \Delta[n]_k$  and  $\tau_k \in \Delta[m]_k$ .  $\Delta[n]_k$  can be identified as  $\mathrm{Hom}_\Delta(k, n)$ . Under this identification, let  $\sigma_k$  correspond to  $\hat{\sigma}_k$ . Given an element  $g \in \mathrm{Hom}(\Delta', \Delta Y'')$ , define  $\hat{g} : \Delta' \times \Delta' \rightarrow Y''$  by  $\hat{g}(\sigma_k \times \tau_k) = \hat{\sigma}_k \times \hat{\tau}_k(g(i_k))$ . Here  $i_k$  is the nondegenerate  $k$ -simplex in  $\Delta[k]$ . Given an element  $f \in \mathrm{Hom}(\Delta' \times \Delta', Y'')$  restrict this map by the diagonal inclusion  $\Delta' \rightarrow \Delta' \times \Delta'$  to get an element of  $\mathrm{Hom}(\Delta', \Delta Y'')$ . It is easy to check that these maps are natural two-sided inverses.

The second statement of the lemma is equivalent to having a pro-isomorphism  $\{\mathrm{Hom}(sk_s \Delta' \times sk_s \Delta', Y'')\} \rightarrow \{\mathrm{Hom}(sk_s \Delta', \Delta Y'')\}$ . Arguments similar to those above show that for each  $s$   $\mathrm{Hom}(sk_s \Delta', \Delta Y'')$  is isomorphic to  $\mathrm{Hom}(sk_s(\Delta' \times \Delta'), Y'')$ . So the proof of the lemma is finished by noting that the inclusion  $\{sk_s(\Delta' \times \Delta')\} \rightarrow \{sk_s \Delta' \times sk_s \Delta'\}$  is a pro-isomorphism of simplicial sets.  $\square$

For  $R^\Delta X'$  this proposition shows that  $\text{Tot } R^\Delta X'$  is isomorphic to  $\text{Tot}(R_\infty X')$  and  $\{\text{Tot}_s(R_s X')\} \rightarrow \{\text{Tot}_s R^\Delta X'\}$  is a pro-isomorphism.

Now we consider the Eilenberg-Moore spectral sequence for the following diagram

$$(5) \quad \begin{array}{ccc} & & Y \\ & & \downarrow f \\ X & \longrightarrow & B \end{array}$$

where  $f$  is a fibration. If  $H_* X, H_* B,$  and  $H_* Y$  are finite type then the codegrees of the cobar construction  $B'$  are finite type. Thus we can apply Corollary 7.7 to see that the Eilenberg-Moore spectral sequence here is pro-converging to the tower  $\{H_* \text{Tot}_s R^\Delta B'\}$ .

Applying Proposition 8.1, we see that  $\{\text{Tot}_s R^\Delta B'\}$  is weakly pro-homotopy equivalent to  $\{\text{Tot}_s^v \text{Tot}_s^h R' B'\}$ . Let the cosimplicial direction in  $B'$  be the vertical direction. Consider  $\{\text{Tot}_s^h R' B'\} = \{R_s B'\}$ . Since each codegree of  $B'$  is a product of copies of  $X, Y,$  and  $B$  by Lemma 5.1 this tower of cosimplicial spaces is weakly pro-homotopy equivalent on each codegree to  $\{B'_s\}$ , where  $B'_s$  is the cobar construction of the following diagram.

$$(6) \quad \begin{array}{ccc} & & R_s Y \\ & & \downarrow R_s f \\ R_s X & \longrightarrow & R_s B \end{array}$$

Thus applying  $\text{Tot}_s$  to these two towers gives a weak pro-homotopy equivalence,  $\{\text{Tot}_s^v \text{Tot}_s^h R' B'\} \rightarrow \{\text{Tot}_s B'_s\}$ . Hence  $\{\text{Tot}_s R^\Delta B'\}$  is weak pro-homotopy equivalent to  $\{\text{Tot}_s B'_s\}$ .

Any cobar construction has the property that  $\text{Tot } B' \simeq \cdots \simeq \text{Tot}_2 B' \simeq \text{Tot}_1 B'$  and  $\text{Tot}_1 B'$  is the homotopy pull-back of the diagram being considered. Let  $M_s$  be the homotopy pull-back of diagram (6). Then  $\text{Tot}_s B'_s$  is homotopy equivalent to  $M_s$  for  $s > 0$ . Hence  $\{\text{Tot}_s R^\Delta B'\}$  is weakly pro-homotopy equivalent to  $\{M_s\}$ . So we can state the following corollary to Corollary 7.7.

**COROLLARY 8.2.** *Let  $B'$  be the cobar construction for diagram (5). If  $H_* X, H_* B,$  and  $H_* Y$  are finite type, then the Eilenberg-Moore spectral sequence for diagram (5) pro-converges to  $\{H_* M_s\}$ . More precisely, there is a sequence of weak pro-homotopy equivalences between  $\{H_* M_s\}$  and  $\{H_* T_s(R \otimes B')\}$ .*

This is a generalization to fibre squares of one of W. Dwyer's exotic convergence results for fibrations [D2, 1.1].

Let  $M_\infty$  be the homotopy pull-back of the following diagram

$$\begin{array}{ccc}
 & R_\infty Y & \\
 & \downarrow R_\infty f & \\
 R_\infty X & \longrightarrow & R_\infty B.
 \end{array}$$

Then  $M_\infty = \varprojlim M_s$ . Thus we can state the following strong convergence corollary to Corollary 7.6.

**COROLLARY 8.3.** *Consider diagram (5). Let  $H_*X$ ,  $H_*B$ , and  $H_*Y$  be finite type. Assume either*

- (a)  $H_*M_\infty$  is finite type, or
- (b)  $\varprojlim H_*M_s$  is finite type.

*Then the Eilenberg-Moore spectral sequence for diagram (5) strongly converges to  $H_*M_\infty$  if and only if  $M_\infty$  is  $p$ -good.*

**9. Relative convergence.** In this section we consider yet another type of convergence question. Here we ask whether a map of cosimplicial spaces which induces an  $E^2$ -isomorphism on the homology spectral sequence gives us any information about the relationship between the total spaces of the cosimplicial spaces.

Let  $f : X' \rightarrow Y'$  be a map of cosimplicial spaces which induces an  $E^r$ -isomorphism on the homology spectral sequences. When  $H_*X^n$  and  $H_*Y^n$  are finite type for each  $n$ , we know that each of these spectral sequences is pro-convergent to  $\{H_*\text{Tot}_s R^\Delta X'\}$  by Corollary 7.7. This shows that the horizontal maps in the following diagram are pro-isomorphisms.

$$\begin{array}{ccccc}
 \{H_*\text{Tot}_s R^\Delta X'\} & \longrightarrow & \{H_*T_s(R \otimes R^\Delta X')\} & \longleftarrow & \{H_*T_s(R \otimes X')\} \\
 \downarrow & & \downarrow & & \downarrow \\
 \{H_*\text{Tot}_s R^\Delta Y'\} & \longrightarrow & \{H_*T_s(R \otimes R^\Delta Y')\} & \longleftarrow & \{H_*T_s(R \otimes Y')\}
 \end{array}$$

The right-hand map is a pro-isomorphism by Lemma 2.2 because  $f$  is an  $E^r$ -isomorphism. Thus we conclude that the left-hand map is also a pro-isomorphism.

By generalizing [BK, III 6.2] to the case of nonconnected spaces, we see that a homology isomorphism  $H_*W \rightarrow H_*Z$  induces a pro-homotopy equivalence  $\{R_s W\} \rightarrow \{R_s Z\}$ . Applying this to the above pro-homology isomorphism gives a pro-homotopy equivalence  $\{R_s \text{Tot}_s R^\Delta X'\} \rightarrow \{R_s \text{Tot}_s R^\Delta Y'\}$ . Each  $\text{Tot}_s R^\Delta X'$  is  $p$ -

nilpotent [B2, 4.6].  $\text{Tot}_s R^\Delta X'$  also has finitely many components by Lemma 7.2 and Lemma 2.6. Hence Lemma 5.2 shows that  $\{\text{Tot}_s R^\Delta X'\}$  is pro-homotopy equivalent to  $\{R_s \text{Tot}_s R^\Delta X'\}$ . The same is true for  $Y'$ . So we conclude that  $\{\text{Tot}_s R^\Delta X'\} \rightarrow \{\text{Tot}_s R^\Delta Y'\}$  is a pro-homotopy equivalence. Because of the finite type assumptions (Lemma 7.2 and Lemma 2.6), each homotopy group of  $\text{Tot}_s R^\Delta X'$  is finite. So  $\varprojlim^1 \pi_i \text{Tot}_s R^\Delta X'$  is zero for each  $i$ . The same is true for  $Y'$ . Thus we conclude that  $\text{Tot} R^\Delta X' \rightarrow \text{Tot} R^\Delta Y'$  is a homotopy equivalence [BK, IX 3.1]. We state this relative convergence result in the following theorem.

**THEOREM 9.1.** *Let  $f : X' \rightarrow Y'$  be a map of cosimplicial spaces which induces an  $E'$ -isomorphism on the homology spectral sequences. If  $H_* X^n$  and  $H_* Y^n$  are finite type for each  $n$ , then the following map is a homotopy equivalence.*

$$\text{Tot} R^\Delta X' \xrightarrow{\simeq} \text{Tot} R^\Delta Y'$$

Using Lemma 7.5, we can apply this theorem to  $R^\Delta X'$  itself to deduce the following corollary.

**COROLLARY 9.2.** *Let  $X'$  be a cosimplicial space. If  $H_* X^n$  is finite type for each  $n$ , then the following map is a homotopy equivalence.*

$$\text{Tot} R^\Delta X' \xrightarrow{\simeq} \text{Tot} R^\Delta R^\Delta X'$$

**10. The  $p$ -resolution of a cosimplicial space.** Because of the exotic convergence results in section 7 and the relative convergence results in 9, we would like to understand  $\text{Tot} R^\Delta X'$  and its relationship to  $\text{Tot} X'$ .

Let the constant cosimplicial space associated to  $X$  be denoted  $c'X$ . Then  $\text{Tot} R^\Delta(c'X)$  is by definition the  $p$ -completion of  $X$ ,  $R_\infty X$ , which was studied in [BK]. So  $R_\infty(\text{Tot} c'X) \simeq \text{Tot} R^\Delta(c'X)$ . In general the relationship between  $\text{Tot} X'$  and  $\text{Tot} R^\Delta X'$  is not this straightforward. The following theorems discuss two special cases in which we can relate these two spaces. First we need the following lemma.

**LEMMA 10.1.** *Let  $R_\infty X'$  be the cosimplicial space with  $(R_\infty X')^n = R_\infty X^n$ . Then  $R_\infty X'$  is a fibrant cosimplicial space.*

*Proof.* To show that  $R_\infty X'$  is fibrant we must show that  $R_\infty X^n \rightarrow M^{n-1}(R_\infty X')$  is a fibration for each  $n$ . This map is the map of total spaces induced by  $R'X^n \rightarrow M^{n-1}(R'X')$  because the total space functor commutes with inverse limit constructions such as the matching space. The  $m$ th codegree of this map of cosimplicial spaces is  $R^m X^n \rightarrow M^{n-1}(R^m X')$ . A fibration of cosimplicial spaces induces a fibration on the total spaces because the category of cosimplicial spaces is a simplicial model category. Thus it is enough to show that  $R'X^n \rightarrow M^{n-1}(R'X')$

is a fibration of cosimplicial spaces. To show this we will use the fact that an epimorphism of group-like cosimplicial spaces is a fibration [BK, X 4.9].

Any choice of basepoint in  $X^n$  makes  $R'X^n$  a group-like cosimplicial space [BK, X 4.10]. Each codegree of  $M^{n-1}(R'X')$  is a simplicial group because it is the inverse limit of a diagram of homomorphisms of simplicial groups. Similarly each cosimplicial operator other than  $d^0$  is a homomorphism. Hence  $M^{n-1}(R'X')$  is group-like. To see that the map between these group-like cosimplicial spaces is an epimorphism it is enough to see that each level is an epimorphism. As noted above the levels of this map are  $R^mX^n \rightarrow M^{n-1}(R^mX')$ . For each  $m$  this map is an epimorphism because  $R^mX'$  is a group-like cosimplicial space [BK, X 4.9].  $\square$

**THEOREM 10.2.** *Let  $X'$  be a cosimplicial space such that each  $X^n$  is  $p$ -complete. Then the following maps are homotopy equivalences.*

$$\text{Tot}\bar{X}' \xrightarrow{\cong} \text{Tot}R^\Delta\bar{X}' \xleftarrow{\cong} \text{Tot}R^\Delta X'$$

*Proof.* Proposition 8.1 shows that  $\text{Tot}R^\Delta\bar{X}'$  is isomorphic to  $\text{Tot}R_\infty\bar{X}'$ . So to show that the first map is a homotopy equivalence we need to show that  $\text{Tot}\bar{X}' \rightarrow \text{Tot}R_\infty\bar{X}'$  is a homotopy equivalence. To do this we consider the map  $\bar{X}' \rightarrow R_\infty\bar{X}'$ . This map is a weak equivalence of cosimplicial spaces because each  $X^n$  is  $p$ -complete. The lemma above shows that  $R_\infty\bar{X}'$  is fibrant. Hence this map is a weak equivalence of fibrant cosimplicial spaces. So it induces a homotopy equivalence on the total spaces by Lemma 2.1.

The functor  $R$  preserves weak equivalences. So since  $X' \rightarrow \bar{X}'$  is a weak equivalence,  $R^\Delta X' \rightarrow R^\Delta\bar{X}'$  is a weak equivalence. By Lemma 7.1 both of these cosimplicial spaces are fibrant. Hence  $\text{Tot}R^\Delta X' \rightarrow \text{Tot}R^\Delta\bar{X}'$  is a homotopy equivalence by Lemma 2.1.  $\square$

We now prove a strong convergence result which generalizes Theorem 6.1 by requiring only that each codegree is  $p$ -complete.

**COROLLARY 10.3.** *Let  $X'$  be a cosimplicial space with  $H_*X^n$  finite type and  $X^n$   $p$ -complete for each  $n$ . Assume either*

- (a)  $H_*\text{Tot}R^\Delta\bar{X}'$  is finite type (equivalently  $H_*\text{Tot}\bar{X}'$  is finite type), or
- (b)  $\varprojlim H_*\text{Tot}_s R^\Delta\bar{X}'$  is finite type (equivalently  $\{H_*T_s(R \otimes \bar{X}')\}$  is pro-constant).

*Then  $\{H_*\text{Tot}\bar{X}'\} \rightarrow \{H_*T_s(R \otimes \bar{X}')\}$  is a pro-isomorphism, i.e., the homology spectral sequence for  $\bar{X}'$  strongly converges, if and only if  $\text{Tot}\bar{X}'$  is  $p$ -good.*

*Proof.* First we should note that the equivalence of the two statements in condition (b) follows from the fact that  $R^\Delta\bar{X}'$  is pro-convergent here. Thus  $\{H_*\text{Tot}_s R^\Delta\bar{X}'\} \rightarrow \{H_*T_s(R \otimes R^\Delta\bar{X}')\}$  is a pro-isomorphism. Lemma 7.5 shows

that these towers are also pro-isomorphic to  $\{H_*T_s(R \otimes \bar{X}^n)\}$ . These towers are pro-finite type by arguments similar to those before the statement of Theorem 6.1. Thus the inverse limit of  $\{H_*\text{Tot}_s R^\Delta \bar{X}^n\}$  is finite type if and only if  $\{H_*T_s(R \otimes \bar{X}^n)\}$  is pro-constant. Using these pro-isomorphisms and Theorem 10.2 this corollary follows easily from Corollary 7.4 above.  $\square$

To identify  $\text{Tot} R^\Delta X^*$  when each  $X^n$  is not  $p$ -complete we must ask that both  $X^*$  and  $R^\Delta X^*$  be strongly convergent.

**THEOREM 10.4** *If  $X^*$  and  $R^\Delta X^*$  are both strongly convergent and  $\text{Tot} R^\Delta X^*$  is  $p$ -good then the following map is a homotopy equivalence of  $p$ -complete spaces.*

$$R_\infty \text{Tot} \bar{X}^* \xrightarrow{\cong} \text{Tot} R^\Delta \bar{X}^*$$

*Proof.* Since  $R^\Delta X^*$  is strongly convergent  $R^\Delta \bar{X}^*$  is also strongly convergent. Thus the horizontal maps in the following diagram are pro-isomorphisms.

$$\begin{array}{ccc} \{H_*\text{Tot} \bar{X}^*\} & \longrightarrow & \{H_*T_s(R \otimes \bar{X}^*)\} \\ \downarrow & & \downarrow \\ \{H_*\text{Tot} R^\Delta \bar{X}^*\} & \longrightarrow & \{H_*T_s(R \otimes R^\Delta \bar{X}^*)\} \end{array}$$

By Lemma 7.5 and Lemma 2.2 the right-hand map is a pro-isomorphism. Thus we conclude that the left-hand map is a pro-isomorphism.  $\text{Tot} R^\Delta \bar{X}^*$  is  $p$ -good and an inverse limit of  $p$ -nilpotent spaces. Hence by [S, 5.3] it is  $p$ -complete. So the map  $\text{Tot} \bar{X}^* \rightarrow \text{Tot} R^\Delta \bar{X}^*$  is a homology isomorphism to a  $p$ -complete space. Thus there is a homotopy equivalence  $R_\infty \text{Tot} \bar{X}^* \rightarrow \text{Tot} R^\Delta \bar{X}^*$  [BK, VII 2.1].  $\square$

Note that if one uses the strong convergence conditions in Theorem 6.1 and Corollary 7.4 then  $\text{Tot} \bar{X}^*$  and  $\text{Tot} R^\Delta \bar{X}^*$  must be  $p$ -good. Also Theorem 6.1 requires that each  $X^n$  is  $p$ -nilpotent. Hence [S, 5.3] implies that  $\text{Tot} \bar{X}^*$  and  $\text{Tot} R^\Delta \bar{X}^*$  are  $p$ -complete. Thus

$$\text{Tot} \bar{X}^* \xrightarrow{\cong} R_\infty(\text{Tot} \bar{X}^*) \xrightarrow{\cong} \text{Tot} R^\Delta \bar{X}^* \xrightarrow{\cong} R_\infty(\text{Tot} R^\Delta \bar{X}^*).$$

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