TOPOLOGICAL EQUIVALENCES FOR DIFFERENTIAL GRADED ALGEBRAS

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ABSTRACT. We investigate the relationship between differential graded algebras (dgas) and topological ring spectra. Every dga C gives rise to an Eilenberg-Mac Lane ring spectrum denoted HC. If HC and HD are weakly equivalent, then we say C and D are topologically equivalent. Quasi-isomorphic dgas are topologically equivalent, but we produce explicit counter-examples of the converse. We also develop an associated notion of topological Morita equivalence using a homotopical version of tilting.

CONTENTS

1. INTRODUCTION

This paper deals with the relationship between differential graded algebras (dgas) and topological ring spectra. DGAs are considered only up to quasi-isomorphism, and ring spectra only up to weak equivalence—both types of equivalence will be denoted $\simeq$ in what follows. Every dga C gives rise to an Eilenberg-MacLane ring spectrum denoted HC (recalled in Section 2.6), and of course if $C \simeq D$ then $HC \simeq HD$. It is somewhat surprising that the converse of this last statement is not true: dgas which are not quasi-isomorphic can give rise to weakly equivalent ring spectra. If $HC \simeq HD$ we will say that the dgas C and D are **topologically equivalent**. Our goal in this paper is to investigate this notion, with examples and applications. [Ke, 3.9] and [S2] give expository accounts of some of this material.

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1.1. **Explicit examples.** In the first sections of the paper we are concerned with producing examples of dgas which are topologically equivalent but not quasi-isomorphic. One simple example turns out to be $C = \mathbb{Z}[e_1; de = 2]/(e^4)$ and $D = \Lambda_{\mathbb{F}_2}(g_2)$. That is to say, C is a truncated polynomial algebra with $de = 2$ and D is an exterior algebra with zero differential. The subscripts on e and g denote the degrees of those elements. To see that these dgas are indeed topologically equivalent, we analyze Postnikov towers and their associated k-invariants. This reduces things to a question about the comparison map between Hochschild cohomology HH^* and topological Hochschild cohomology THH^*, which one can resolve by referring to calculations in the literature.

Similar—but more complicated—examples exist with $\mathbb{F}_2$ replaced by $\mathbb{F}_p$. It is an interesting feature of this subject that as p grows the complexity of the examples becomes more and more intricate.

1.2. **Model categories.** The above material has an interesting application to model categories. Recall that a Quillen equivalence between two model categories is a pair of adjoint functors satisfying certain axioms with respect to the cofibrations, fibrations, and weak equivalences. Two model categories are called Quillen equivalent if they can be connected by a zig-zag of such adjoint pairs. Quillen equivalent model categories represent the same 'underlying homotopy theory'.

By an **additive model category** we mean one whose underlying category is additive and where the additive structure behaves well with respect to 'higher homotopies'; a precise definition is given in [DS2, Section 2]. So this excludes the model categories of simplicial sets and topological spaces but includes most model categories arising from homological algebra. Two additive model categories are **additively Quillen equivalent** if they can be connected by a zig-zag of Quillen equivalences in which all the intermediate steps are additive. The following is a strange and interesting fact:

- *It is possible for two additive model categories to be Quillen equivalent but not additively Quillen equivalent.*

One might paraphrase this by saying that there are two 'algebraic' model categories which have the same underlying homotopy theory, but where the equivalence cannot be seen using only algebra! Any zig-zag of Quillen equivalences between the two must necessarily pass through a non-additive model category.

In Section 8 we present an example demonstrating the above possibility. It comes directly from the two dgas C and D we wrote down in (1.1). If A is any dga, the category of differential graded A-modules (abbreviated as just 'A-modules' from now on) has a model structure in which the weak equivalences are quasi-isomorphisms and the fibrations are surjections. We show that the model categories of C-modules and D-modules are Quillen equivalent but not additively Quillen equivalent.

1.3. **Topological tilting theory for dgas.** The above example with model categories is actually an application of a more general theory. One may ask the following question: Given two dgas A and B, when are the categories of A-modules and B-modules Quillen equivalent? We give a complete answer in terms of a homotopical tilting theory for dgas, and this involves topological equivalence.

Recall from Morita theory that two rings R and S have equivalent module categories if and only if there is a finitely-generated S-projective P such that P is a strong generator and the endomorphism ring $\mathrm{Hom}_S(P, P)$ is isomorphic to R. Rickard [Ri] developed an analogous criterion for when the derived categories $\mathcal{D}(R)$ and $\mathcal{D}(S)$ are equivalent triangulated categories, and this was extended in [DS1, 4.2] to the model category level. Explicitly, the model categories of chain complexes Ch_R and Ch_S are Quillen equivalent if and only if there is a bounded complex of finitely-generated S-projectives P_* which is a weak generator for the derived category $\mathcal{D}(S)$, and whose endomorphism dga $\mathrm{Hom}_S(P, P)$ is quasi-isomorphic to R (regarded as a dga concentrated in dimension zero).

We wish to take this last result and allow R and S to be dgas rather than just rings. Almost the same theorem is true, but topological equivalence enters the picture:

Theorem 1.4. *Let C and D be two dgas. The model categories of C- and D-modules are Quillen equivalent if and only if there is a cofibrant and fibrant representative P of a compact generator in $\mathcal{D}(C)$ whose endomorphism dga $\mathrm{Hom}_C(P, P)$ is topologically equivalent to D.*

See Definition 7.1 for the definition of a compact generator. If $\mathcal{D}(C)$ has a compact generator, then a cofibrant and fibrant C-module representing this generator always exists.

If one takes Theorem 1.4 and replaces 'topologically equivalent' with 'quasi-isomorphic', the resulting statement is false. For an example, take C and D to be the two dgas already mentioned in Section 1.1. The model categories of C- and D-modules are Quillen equivalent since they only depend on HC and HD, but it is easy to show (see Section 8) that no D-module can have C as its endomorphism dga.

We have the following parallel of the above theorem, however:

Theorem 1.5. *Let C and D be two dgas. The model categories of C- and D-modules are **additively** Quillen equivalent if and only if there is a cofibrant and fibrant representative P of a compact generator in $\mathcal{D}(C)$ such that $\mathrm{Hom}_C(P, P)$ is quasi-isomorphic to D.*

1.6. **Topological equivalence over fields.** We do not know a general method for deciding whether two given dgas are topologically equivalent or not. The examples of topological equivalence known to us all make crucial use of dealing with dgas over $\mathbb{Z}$. It would be interesting to know if there exist nontrivial examples of topological equivalence for dgas defined over a field. Here is one negative result along these lines, whose proof is given in Section 5.7.

Proposition 1.7. *If C and D are both $\mathbb{Q}$-dgas, then they are topologically equivalent if and only if they are quasi-isomorphic.*

1.8. **Organization.** The main results of interest in this paper—described above—are contained in Sections 5, 7, and 8. Readers who are impatient can jump straight to those sections.

Sections 2 through 4 establish background on dgas and ring spectra needed for the examples in Section 5. This background material includes Postnikov sections, k-invariants, and the role of Hochschild and topological Hochschild cohomologies.

The tilting theory results are given in Section 7. To prove these, one needs to use certain invariants of stable model categories—namely, the homotopy endomorphism ring spectra of [D2]. These invariants are defined very abstractly and so are difficult to compute, but in Section 6 we state some auxiliary results simplifying things in the case of model categories enriched over $Ch_{\mathbb{Z}}$. The proofs of these results are rather technical and appear in [DS2].

1.9. **Notation and terminology.** If $\mathcal{M}$ and $\mathcal{N}$ are model categories, a Quillen pair $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ will also be referred to as a **Quillen map** $L\colon \mathcal{M} \to \mathcal{N}$. The terms 'strong monoidal-' and 'weak monoidal Quillen pair' will be used often—they are defined in [SS3, 3.6]. A **strong monoidal Quillen pair** is basically a Quillen pair between monoidal model categories where L is strong monoidal. Finally, if $\mathcal{C}$ is a category we write $\mathcal{C}(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Throughout the paper, for a symmetric spectrum X the notation $\pi_* X$ denotes the derived homotopy groups; that is, one first replaces X by a fibrant spectrum and then evaluates π_*.

2. Background on dgas and ring spectra

In this section we review model category structures on dgas and ring spectra. We also recall the construction which associates to every dga a corresponding Eilenberg-MacLane ring spectrum.

2.1. **DGAs.** If k is a commutative ring, let Ch_k denote the category of (unbounded) chain complexes of k-modules. Just to be clear, we are grading things so that the differential has the form $d\colon C_n \to C_{n-1}$. Recall that Ch_k has a model category structure—called the 'projective' model structure—in which the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections [Ho1, 2.3.11]. The tensor product of chain complexes makes this into a symmetric monoidal model category in the sense of [Ho1, 4.2.6].

By a ***k*-dga** we mean an object $X \in Ch_k$ together with maps $k[0] \to X$ and $X \otimes X \to X$ giving an associative and unital pairing. Here $k[0]$ is the complex consisting of a single k concentrated in dimension 0. Note that we only require associativity here, not commutativity.

By [SS1, 4.1(2)], a model category structure on k-dgas can be lifted from the projective model structure on Ch_k. The weak equivalences of k-dgas are again the quasi-isomorphisms and the fibrations are the surjections. The cofibrations are then determined by the left lifting property with respect to the acyclic fibrations. A more explicit description of the cofibrations comes from recognizing this model structure as a cofibrantly generated model category [Ho1, 2.1.17]. Let $k(S^n)$ be the free k-algebra on one generator in degree n with zero differential. Let $k(D^{n+1})$ be the free k-algebra on two generators x and y with $|x| = n$ and $|y| = n+1$ such that $dx = 0$ and $dy = x$. The generating cofibrations are the inclusions $i_n\colon\ k(S^n) \to k(D^{n+1})$ and the generating acyclic cofibrations are the maps $j_n\colon\ 0 \to k(D^{n+1})$. These generating maps are constructed from the generating maps for Ch_k [Ho1, 2.3.3] by applying the free tensor algebra functor T_k. Given C in Ch_k, $T_k(C) = \oplus_{n\geq 0} C^{\otimes n}$ where $C^{\otimes 0} = k[0]$ and all tensor products are over k.

Remark 2.2. Cofibrant replacements of k-dgas play an important role in what follows. There is a functorial cofibrant replacement arising from the cofibrantly generated structure [SS1], but this gives a very large model. We sketch a construction of a smaller cofibrant replacement which is useful in calculations. Suppose given C in $k{-}\mathcal{DGA}$ with $H_i C = 0$ for $i < 0$. Choose generators of $H_* C$ as a k-algebra. Let G be the free graded k-module on the given generators and let $T_1 C = T_k(G)$

be the associated free k-dga with zero differential. Define $f_1\colon T_1C \to C$ by sending each generator in G to a chosen cycle representing the associated generator in H_*C. The induced map $(f_1)_*$ in homology is surjective. Let n be the smallest degree in which $(f_1)_*$ has a kernel, and pick a set of k-module generators for the kernel in this dimension. For each chosen generator there is an associated cycle in T_1C and thus a map $k(S^n) \to T_1(C)$. The pushout $k(D^{n+1}) \leftarrow k(S^n) \to T_1(C)$ has the effect of killing the associated element in $H_n[T_1(C)]$. Let T_2C be the pushout of

$$\coprod k(D^{n+1}) \leftarrow \coprod k(S^n) \to T_1C$$

where the coproduct runs over the chosen generators. (Note that the coproduct is taken in the category $k-\mathcal{D}GA$, and so is complicated—it's analogous to an amalgamated product of groups.) Since the elements being killed here are in the kernel of $(f_1)_*$, each map $k(S^n) \to T_1(C) \to C$ extends to a map $k(D^{n+1}) \to C$. Thus, there is a map from the pushout $f_2\colon T_2C \to C$. The induced map $(f_2)_*$ is an isomorphism up through degree n, and now one repeats the process in degree $n+1$. The colimit of the resulting T_iC's is a cofibrant replacement $T_\infty C \to C$. (Note that $T_\infty C \to C$ is not a fibration, however.)

Example 2.3. By way of illustration, let $C = \mathbb{Z}/2$ considered as a $\mathbb{Z}$-dga (concentrated in degree 0). We may take $T_1C = \mathbb{Z}$. The element 2 must now be killed in homology, so we attach a free generator e in degree 1 to kill it. That is, we form the pushout of a diagram

$$\mathbb{Z}(D^1) \leftarrow \mathbb{Z}(S^0) \to T_1C.$$

The result is $T_2(C) = \mathbb{Z}[e; de = 2]$. The map $T_2C \to \mathbb{Z}$ is an isomorphism on H_0 and H_1, but on H_2 we have a kernel (generated by e^2). After forming the appropriate pushout we have $T_3C = \mathbb{Z}\langle e_1, f_3; de = 2, df = e^2\rangle$ (forgetting the differentials, this is a tensor algebra on e and f).

Next note that $H_3(T_3C) = 0$, but $H_4(T_3C)$ is nonzero—it's generated by $ef + fe$. So we adjoin an element g_5 with $dg = ef + fe$. Now T_4C is a tensor algebra on e, f, and g, with certain differentials. One next looks at the homology in dimension 5, and continues. Clearly this process gets very cumbersome as one goes higher and higher in the resolution.

2.4. **Homotopy classes.** Suppose that C and D are k-dgas, and that C is cofibrant. To compute maps in the homotopy category $\mathrm{Ho}\,(k-\mathcal{D}GA)(C, D)$, one may either use a cylinder object for C or a path object for D [Ho1, 1.2.4, 1.2.10]. In the case of general dgas, both are somewhat complicated. For the very special situations that arise in this paper, however, there is a simple method using path objects.

Assume $C_i = 0$ for $i < 0$, C_0 is generated by 1 as a k-algebra, and that $D = k \oplus M$ for some $M \in Ch_k$. Here D is the dga obtained by adjoining M to k as a square-zero ideal. Let I denote the chain complex $k \to k^2$ concentrated in degrees 1 and 0, where $d(a) = (a, -a)$. Recall that $\mathrm{Hom}_k(I, M)$ is a path object for M in Ch_k, and let PD denote the square-zero extension $k \oplus \mathrm{Hom}_k(I, M)$. This is readily seen to be a path object for D in $k-\mathcal{D}GA$. It is not a *good* path object, however, as the map $PD \to D \times D$ is not surjective in degree 0. Despite this fact, it is still true that $\mathrm{Ho}\,(k-\mathcal{D}GA)(C, D)$ is the coequalizer of $k-\mathcal{D}GA(C, PD) \rightrightarrows k-\mathcal{D}GA(C, D)$; this is a simple argument using that all maps of dgas with domain C coincide on C_0.

2.5. **Ring spectra.** There are, of course, different settings in which one can study ring spectra. We will work with the category of symmetric spectra Sp^Σ of [HSS] with its symmetric monoidal smash product $\wedge$ and unit S. A **ring spectrum** is just an S-algebra—that is to say, it is a spectrum R together with a unit $S \to R$ and an associative and unital pairing $R \wedge R \to R$. The category $S - \mathcal{A}lg$ has a model structure in which a map is a weak equivalence or fibration precisely if it is so when regarded as a map of underlying spectra. See [HSS, 5.4.3].

The forgetful functor $S - \mathcal{A}lg \to Sp^\Sigma$ has a left adjoint T. So for any spectrum E there is a 'free ring spectrum built from E', denoted $T(E)$. The cofibrations in $S - \mathcal{A}lg$ are generated (via retracts, cobase-change, and transfinite composition) from those of the form $T(A) \to T(B)$ where $A \to B$ ranges over the generating cofibrations of Sp^Σ. Recall that for Sp^Σ these generators are just the maps $\Sigma^\infty(\partial\Delta^n) \to \Sigma^\infty(\Delta^n)$ and their desuspensions. The situation therefore exactly parallels that of dgas.

2.6. **Eilenberg-MacLane ring spectra.** Any dga A gives rise to a ring spectrum HA called the **Eilenberg-MacLane ring spectrum associated to** A. This can be constructed functorially, and has the property that if $A \to B$ is a quasi-isomorphism then $HA \to HB$ is a weak equivalence. It is also the case that H preserves homotopy limits.

Unfortunately, giving a precise construction of $H(-)$ seems to require a morass of machinery. This is accomplished in [S1]. We will give a brief summary, but the reader should note that these details can largely be ignored for the rest of the paper.

Let ch_+ be the category of non-negatively graded chain complexes. On this category one can form symmetric spectra based on the object $\mathbb{Z}[1]$, as in [Ho2]; call the corresponding category $Sp^\Sigma(\mathrm{ch}_+)$. Similarly one can form symmetric spectra based on simplicial abelian groups and the object $\widetilde{\mathbb{Z}}S^1$; call this category $Sp^\Sigma(s\mathcal{A}b)$.

There are two Quillen equivalences (with left adjoints written on top)

$$Sp^\Sigma(\mathrm{ch}_+) \underset{R}{\overset{D}{\rightleftarrows}} Ch_{\mathbb{Z}} \qquad \text{and} \qquad Sp^\Sigma(\mathrm{ch}_+) \underset{\nu}{\overset{L}{\rightleftarrows}} Sp^\Sigma(s\mathcal{A}b) \tag{2.7}$$

in which (D, R) is strong monoidal and (L, ν) is weak monoidal (the functor ν is denoted $\phi^* N$ in [S1]). See [SS3, 3.6] for the terms 'strong monoidal' and 'weak monoidal'. By the work in [S1] the above functors induce Quillen equivalences between the corresponding model categories of ring objects (or monoids). Thus we have adjoint pairs

$$D\colon \mathcal{R}ing(Sp^\Sigma(\mathrm{ch}_+)) \rightleftarrows \mathcal{R}ing(Ch_{\mathbb{Z}})\colon R$$
$$L^{mon}\colon \mathcal{R}ing(Sp^\Sigma(\mathrm{ch}_+)) \rightleftarrows \mathcal{R}ing(Sp^\Sigma(s\mathcal{A}b))\colon \nu.$$

In the first case the functors D and R are just the restriction of those in (2.7), as these were strong monoidal. But in the second case only the right adjoint is restricted from (2.7); the left adjoint is more complicated. See [SS3, 3.3] for a complete description of L^{mon}.

Finally, the Quillen pair $F\colon s\mathcal{S}et \rightleftarrows s\mathcal{A}b\colon U$ (where U is the forgetful functor) induces a strong monoidal Quillen pair $F\colon Sp^\Sigma \rightleftarrows Sp^\Sigma(s\mathcal{A}b)\colon U$. Since the unit in $Sp^\Sigma(s\mathcal{A}b)$ forgets to the symmetric spectrum $H\mathbb{Z}$, this Quillen pair factors through a strong monoidal Quillen equivalence $Z\colon H\mathbb{Z}\text{-Mod} \rightleftarrows Sp^\Sigma(s\mathcal{A}b)\colon U'$ by [S1, 2.5].

Therefore, by [SS3, 3.2], there is a Quillen pair

$$F\colon S-\mathcal{A}lg \rightleftarrows \mathcal{R}ing[Sp^{\Sigma}(s\mathcal{A}b)]\colon U$$

which factors through the Quillen equivalence

$$Z\colon H\mathbb{Z}-\mathcal{A}lg \rightleftarrows \mathcal{R}ing[Sp^{\Sigma}(s\mathcal{A}b)]\colon U'.$$

Let $\underline{R}$, $\underline{D}$, etc. denote the derived functors of R and D—that is, the induced functors on homotopy categories. If A is a dga, we then define

$$HA = \underline{U}\big[\underline{L}^{mon}(\underline{R}A)\big].$$

This is a ring spectrum with the desired properties. To see that H preserves homotopy limits, for instance, note that both L^{mon} and R have this property because they are functors in a Quillen equivalence; likewise U has this property because it is a right Quillen functor.

3. Postnikov sections and k-invariants for dgas

Fix a commutative ring k. In this section we work in the model category k–$\mathcal{D}GA$, and describe a process for understanding the quasi-isomorphism type of a dga. This involves building the dga from the ground up, one degree at a time, by looking at its Postnikov sections. The difference between successive Postnikov sections is measured by a homotopical extension, usually called a k-invariant (unfortunately the 'k' in 'k-invariant' has no relation to the commutative ring k!) This k-invariant naturally lives in a certain Hochschild cohomology group. In Example 3.15 we use these tools to classify all dgas over $\mathbb{Z}$ whose homology is $\Lambda_{\mathbb{F}_p}(g_n)$.

3.1. Postnikov sections.

Definition 3.2. *If C is a dga and $n \geq 0$, an **nth Postnikov section** of C is a dga X together with a map $C \to X$ such that*

(i) $H_i(X) = 0$ for $i > n$, and
(ii) $H_i(C) \to H_i(X)$ is an isomorphism for $i \leq n$.

Fix an $n \geq 0$. We are able to construct Postnikov sections only when C is a **non-negative** dga—meaning that $C_i = 0$ when $i < 0$. A functorial nth Postnikov section can be obtained by setting

$$[\mathbb{P}_n C]_i = \begin{cases} C_i & \text{if } i < n, \\ C_n/\operatorname{Im}(C_{n+1}) & \text{if } i = n, \text{ and} \\ 0 & \text{if } i > n. \end{cases}$$

This construction is sometimes inconvenient in that the map $C \to \mathbb{P}_n C$ is not a cofibration. To remedy this, there is an alternative construction using the small object argument. Given any cycle z of degree $n+1$, there is a unique map $k(S^{n+1}) \to C$ sending the generator in degree $n+1$ to z. We can construct the pushout of $k(D^{n+2}) \leftarrow k(S^{n+1}) \to C$, which has the effect of killing z in homology. Let $L_{n+1}C$ be the pushout

$$\coprod k(D^{n+2}) \longleftarrow \coprod k(S^{n+1}) \longrightarrow C$$

where the coproduct runs over all cycles in degree $n+1$. Define P_nC to be the colimit of the sequence

$$C \to L_{n+1}C \to L_{n+2}L_{n+1}C \to L_{n+3}L_{n+2}L_{n+1}C \to \cdots$$

It is simple to check that this is indeed another functorial nth Postnikov section, and $C \to P_nC$ is a cofibration. Note as well that there is a natural projection $P_nC \to \mathbb{P}_nC$, which is a quasi-isomorphism.

Proposition 3.3. *For any non-negative dga C and any nth Postnikov section X, there is a quasi-isomorphism $P_nC \to X$.*

Proof. The quasi-isomorphism can be constructed directly using the description of the functors L_i. □

It follows from the above proposition that any two nth Postnikov sections for C are quasi-isomorphic.

Note that there are canonical maps $P_{n+1}C \to P_nC$ compatible with the co-augmentations $C \to P_iC$. The sequence $\cdots \to P_{n+1}C \to P_nC \to \cdots \to P_0C$ is called the **Postnikov tower** for C.

3.4. **k-invariants.** Let C be a dga and let M be a C-bimodule—i.e., a $(C \otimes_k C^{op})$-module. By $C \vee M$ we mean the square-zero extension of C by M; it is the dga whose underlying chain complex is $C \oplus M$ and whose algebra structure is the obvious one induced from the bimodule structure (and where $m \cdot m' = 0$ for any $m, m' \in M$).

Assume C is non-negative. Then there is a natural map $P_0(C) \to H_0(C)$, and it is a quasi-isomorphism. Since $H_{n+1}(C)$ is a bimodule over $H_0(C)$, it becomes a bimodule over each P_nC by restriction through $P_nC \to P_0C \to H_0(C)$. So we can look at the square-zero extension $P_nC \vee \Sigma^{n+2}H_{n+1}(C)$

There is a canonical map of dgas $\gamma\colon P_nC \to \mathbb{P}_nC \vee \Sigma^{n+2}[H_{n+1}C]$ defined by letting it be the identity in dimensions smaller than n, the natural projection in dimension n, and the zero map in dimension $n+1$ and all dimensions larger than $n+2$. In dimension $n+2$ it can be described as follows. The map is zero on C_{n+2}, and if $x \in [P_nC]_{n+2}$ was adjoined to kill the cycle $z \in C_{n+1}$, then x is mapped to the class of z in $\Sigma^{n+2}[H_{n+1}(C)]$. A little checking shows γ is a well-defined map of dgas.

Let $\mathrm{Ho}\,(k{-}\mathcal{D}GA_{/\mathbb{P}_nC})$ denote the homotopy category of k-dgas augmented over $\mathbb{P}_nC$, and let $\alpha_n \in \mathrm{Ho}\,(k{-}\mathcal{D}GA_{/\mathbb{P}_nC})(P_nC, \mathbb{P}_nC \vee \Sigma^{n+2}[H_{n+1}C])$ be the image of the map γ. Then α_n is called the ***n*th *k*-invariant** of C. One can check that it depends only on the quasi-isomorphism type of $P_{n+1}C$. Moreover, the homotopy type of $P_{n+1}C$ can be recovered from α_n. This is shown by the following result, since of course the homotopy fiber of γ only depends on its homotopy class:

Proposition 3.5. $P_{n+1}C \to P_nC \xrightarrow{\gamma} \mathbb{P}_nC \vee \Sigma^{n+2}[H_{n+1}C]$ *is a homotopy fiber sequence in $Ho\,(k{-}\mathcal{D}GA_{/\mathbb{P}_nC})$.*

Proof. The result may be rephrased as saying that $P_{n+1}C$ is weakly equivalent to the homotopy pullback (in the category of dgas) of the diagram

$$P_nC \longrightarrow \mathbb{P}_nC \vee \Sigma^{n+2}[H_{n+1}C] \longleftarrow \mathbb{P}_nC$$

where the right map is the obvious inclusion. Note also that the left map is a fibration, by construction, and so the pullback and homotopy pullback are equivalent. One readily sees that there is a natural map from $P_{n+1}C$ to this pullback, and that it is a quasi-isomorphism. □

Remark 3.6. As $k-\mathcal{D}GA$ is a right proper model category, it follows that for any $X \xrightarrow{\sim} Y$ the induced Quillen map $k-\mathcal{D}GA_{/X} \to k-\mathcal{D}GA_{/Y}$ is a Quillen equivalence. The quasi-isomorphism $P_nC \to \mathbb{P}_nC$ therefore allows us to identify the set $\mathrm{Ho}\,(k-\mathcal{D}GA_{/\mathbb{P}_nC})(P_nC, \mathbb{P}_nC \vee \Sigma^{n+2}[H_{n+1}C])$ with

$$\mathrm{Ho}\,(k-\mathcal{D}GA_{/P_nC})(P_nC, P_nC \vee \Sigma^{n+2}[H_{n+1}C]).$$

3.7. **Classifying extensions.**

Definition 3.8. *Let C be a non-negative k-dga such that $H_i(C) = 0$ for $i > n$. A **Postnikov $(n+1)$-extension** of C is a k-dga X such that $P_{n+1}X \simeq X$ together with a map of k-dgas $f\colon X \to C$ which yields an isomorphism on $H_i(-)$ for $i \le n$. A map $(X, f) \to (Y, g)$ between Postnikov $(n+1)$-extensions is defined to be a quasi-isomorphism $X \to Y$ compatible with f and g.*

Proposition 3.9. *Assume C is non-negative and that $P_nC \simeq C$. Fix an $H_0(C)$-bimodule M. Consider the category whose objects consist of Postnikov $(n+1)$-extensions (X, f) together with an isomorphism of $H_0(C)$-bimodules $\theta\colon H_{n+1}(X) \to M$. A map from (X, f, θ) to (Y, g, σ) is a quasi-isomorphism $X \to Y$ compatible with the other data. Then the connected components of this category are in bijective correspondence with the set of homotopy classes $Ho\,(k-\mathcal{D}GA_{/C})(C, C \vee \Sigma^{n+2}M)$.*

In the context of the above result, the general problem one would like to be able to solve is to classify all Postnikov $(n+1)$-extensions X of C having $H_{n+1}X \cong M$. It is not quite true that the set of quasi-isomorphism types of all such X is in bijective correspondence with $\mathrm{Ho}\,(k-\mathcal{D}GA_{/C})(C, C \vee \Sigma^{n+2}M)$. Different k-invariants can nevertheless lead to quasi-isomorphic dgas X. To see this, note that if $h\colon M \to M$ is an automorphism of H_0C-bimodules then any k-invariant can be 'twisted' by this automorphism. The homotopy fibers of the original and twisted k-invariants will be quasi-isomorphic, however—the only thing that is different about them is the prescribed isomorphism between their H_{n+1} and M. This is why such an isomorphism must be built into the category appearing in the proposition.

Proof of Proposition 3.9. Let $\mathcal{A}$ be the category described in the statement of the proposition, and let $T = \mathrm{Ho}\,(k-\mathcal{D}GA_{/C})(C, C \vee \Sigma^{n+2}M)$.

Suppose $X \to C$ is a Postnikov $(n+1)$-extension of C, and $\theta\colon H_{n+1}X \to M$ is an isomorphism of $H_0(C)$-bimodules. As above, construct the map $\gamma\colon P_nX \to \mathbb{P}_nX \vee \Sigma^{n+2}[H_{n+1}X]$. The map $X \to C$ induces quasi-isomorphisms $P_nX \to P_nC$ and $\mathbb{P}_nX \to \mathbb{P}_nC$. Note that the maps $C \to P_nC$ and $C \to \mathbb{P}_nC$ are quasi-isomorphisms as well. These, together with θ, allow us to identify γ with a map in the homotopy category $C \to C \vee \Sigma^{n+2}M$. One checks that this gives a well-defined map $\pi_0\mathcal{A} \to T$.

Now suppose $\alpha \in T$. Let $\tilde{C} \to C$ be a cofibrant-replacement, and let $\tilde{C} \to C \vee \Sigma^{n+2}M$ be any map representing α. Let X be the homotopy pullback of $\tilde{C} \to C \vee \Sigma^{n+2}M \leftarrow C$ where the second map is the obvious inclusion. The composition $X \to \tilde{C} \to C$ makes X into a Postnikov $(n+1)$-extension of C, and the long exact sequence for the homology of a homotopy pullback gives us an isomorphism $\theta\colon H_{n+1}X \to M$. One checks that this defines a map $T \to \pi_0\mathcal{A}$. With some trouble one can verify that this is inverse to the previous map $\pi_0\mathcal{A} \to T$. (Remark 3.11 below suggests a better, and more complete, proof). □

The following corollary of the above proposition is immediate:

Corollary 3.10. *Let C and M be as in the above proposition. Let G be the group of $H_0(C)$-bimodule automorphisms of M. Let S be the quotient set of G acting* *on* $Ho\,(k-\mathcal{D}GA_{/C})(C, C\vee\Sigma^{n+2}M)$*. Consider the category of Postnikov* $(n+1)$*-extensions* X *of* C *which satisfy* $H_{n+1}X\cong M$ *as* $H_0(C)$*-bimodules (but where no prescribed choice of isomorphism is given). Then* S *is in bijective correspondence with the connected components of this category.*

Proof. Let $\mathcal{A}$ be the category described in Proposition 3.9 and let $\mathcal{B}$ be the category described in the statement of the corollary. There is clearly a surjective map $\pi_0(\mathcal{A})/G\to\pi_0(\mathcal{B})$. Injectivity is a very simple exercise. □

Remark 3.11. A more complete proof of the above proposition and corollary can be obtained by following the methods of [DS3]. That paper takes place in the setting of ring spectra, but all the arguments adapt verbatim. Alternatively, using [S1] one can consider k-dgas as Hk-algebra spectra—so from that perspective the above results are actually special cases of those from [DS3, 8.1].

Remark 3.12. The above material can be applied to any connective dga C (that is, one where $H_k(C)=0$ for $k<0$). Such a dga is always quasi-isomorphic to a non-negative dga QC. One gets k-invariants in the set

$$\mathrm{Ho}\,(k-\mathcal{D}GA_{/P_n(QC)})\big(P_n(QC), P_n(QC)\vee\Sigma^{n+2}H_{n+1}(QC)\big).$$

Example 3.13. Let C be a dga over $\mathbb{Z}$ with $H_*(C)$ equal to an exterior algebra over $\mathbb{F}_p$ on a generator in degree 2. What are the possibilities for C? We know that $P_1(C)\simeq\mathbb{F}_p$ and $P_2(C)\simeq C$. We therefore have to analyze the single homotopy fiber sequence $P_2C\to\mathbb{F}_p\to\mathbb{F}_p\vee\Sigma^3\mathbb{F}_p$. What are the possibilities for the k-invariant in this sequence? One has to remember here that $\mathbb{F}_p$ is not cofibrant as a dga over $\mathbb{Z}$, and so one must work with a cofibrant replacement.

The first few degrees of a cofibrant replacement for $\mathbb{F}_p$ look like

$$\cdots\longrightarrow\mathbb{Z}e^3\oplus\mathbb{Z}f\xrightarrow{(p,1)}\mathbb{Z}e^2\xrightarrow{0}\mathbb{Z}e\xrightarrow{p}\mathbb{Z}.1$$

These symbols mean $d(e)=p$ (which implies $d(e^n)=pe^{n-1}$ when n is odd and $d(e^n)=0$ when n is even) and $d(f)=e^2$. We are interested in maps from this dga to the simpler dga

$$0\longrightarrow(\mathbb{Z}/p).g\longrightarrow 0\longrightarrow 0\longrightarrow(\mathbb{Z}/p).1$$

(an exterior algebra with a class in degree 3, and zero differential). The possibilities for such maps are clear: e must be sent to 0, and f can be sent to a (possibly zero) multiple of g. One finds that $\mathrm{Ho}\,(\mathbb{Z}-\mathcal{D}GA_{/\mathbb{F}_p})(\mathbb{F}_p,\mathbb{F}_p\vee\Sigma^3\mathbb{F}_p)\cong\mathbb{Z}/p$; this requires an analysis of homotopies, but using (2.4) one readily sees that all homotopies are constant. Now, the group of automorphisms of $\mathbb{F}_p$ as an $\mathbb{F}_p$-module is just $(\mathbb{Z}/p)^*$, and there are precisely two orbits of $\mathbb{Z}/p$ under this group action. By Corollary 3.10, we see that there are precisely two quasi-isomorphism types of $\mathbb{Z}$-dgas having homology algebra $\Lambda_{\mathbb{F}_p}(g_2)$.

We can find these two dgas explicitly by constructing the appropriate homotopy pullbacks. When the k-invariant has $f\mapsto 0$ one finds that C is just an exterior algebra with zero differential (the k-invariant $\mathbb{F}_p\to\mathbb{F}_p\vee\Sigma^3\mathbb{F}_p$ is just the obvious inclusion). When the k-invariant has $f\mapsto g$ one has that C is quasi-isomorphic to the dga $\mathbb{Z}[e; de=p]/(e^4)$.

3.14. **Hochschild cohomology.** We want to extend Example 3.13 and classify all dgas whose homology algebra is $\Lambda_{\mathbb{F}_p}(g_n)$. A problem arises, in that one has to compute a cofibrant-replacement of $\mathbb{F}_p$ (as a $\mathbb{Z}$-dga) up to dimension $n+1$. Such a cofibrant-replacement becomes very large in high dimensions. Hochschild cohomology gives a way around this issue, which we now recall.

One possible reference for the material in this section is [L, Section 2]. Lazarev works in the context of ring spectra, but all his results and proofs work exactly the same for dgas. Looked at differently, k-dgas are the same as ring spectra which are Hk-algebras by [S1]—so the results below are just special cases of Lazarev's, where the ground ring is Hk.

If C is a k-dga, let $\Omega_{C/k}$ denote the homotopy fiber of the multiplication map $C \otimes_k^L C \to C$. From now on we will just write $\otimes_k$ instead of $\otimes_k^L$; but it is important to never forget that all tensors are now *derived* tensors. We also write Hom rather than RHom. If M is a $C \otimes_k C^{op}$-module, there is an induced homotopy fiber sequence

$$\mathrm{Hom}_{C\otimes_k C^{op}}(C, M) \to \mathrm{Hom}_{C\otimes_k C^{op}}(C \otimes_k C^{op}, M) \to \mathrm{Hom}_{C\otimes_k C^{op}}(\Omega_{C/k}, M).$$

The term in the middle may be canonically identified with M. One typically makes the following definitions:

$$\mathbf{Der}_k(C, M) = \mathrm{Hom}_{C\otimes_k C^{op}}(\Omega_{C/k}, M), \qquad \mathbf{HH}_k(C, M) = \mathrm{Hom}_{C\otimes_k C^{op}}(C, M)$$

$$\mathrm{Der}_k^n(C, M) = H_{-n}[\mathbf{Der}_k(C, M)], \qquad \text{and} \qquad \mathrm{HH}_k^n(C, M) = H_{-n}[\mathbf{HH}_k(C, M)].$$

Sometimes we will omit the k subscripts when the ground ring is understood.

The homotopy fiber sequence $\mathbf{HH}(C, M) \to M \to \mathbf{Der}(C, M)$ gives a long exact sequence of the form

$$\cdots \to \mathrm{HH}^n(C, M) \to H_{-n}(M) \to \mathrm{Der}^n(C, M) \to HH^{n+1}(C, M) \to \cdots$$

We will mostly be interested in applying this when M is concentrated in a single dimension r, in which case $\mathrm{Der}^*(C, M) \cong \mathrm{HH}^{*+1}(C, M)$ for $* \notin \{-r, -r-1\}$.

Finally, in order to connect all this with the classification of dgas, one can prove that there is an isomorphism

$$\mathrm{Der}^n(C, M) \cong \mathrm{Ho}\,(k-\mathcal{DGA}_{/C})(C, C \vee \Sigma^n M).$$

The proof of this isomorphism in [L] seems to contain gaps; we are very grateful to Mike Mandell for showing us a complete proof [M].

Example 3.15. We'll use the above machinery to determine all dgas C over $\mathbb{Z}$ such that $H_*(C) \cong \Lambda_{\mathbb{F}_p}(g_n)$ (an exterior algebra on a class of degree n). Such a dga has $P_{n-1}(C) \simeq \mathbb{F}_p$ and $P_n C \simeq C$, so we have the homotopy fiber sequence $C \to \mathbb{F}_p \to \mathbb{F}_p \vee \Sigma^{n+1}\mathbb{F}_p$. We need to understand the possibilities for the second map.

The above observations give us isomorphisms

$$\mathrm{Ho}\,(\mathbb{Z} - \mathcal{DGA}_{/\mathbb{F}_p})(\mathbb{F}_p, \mathbb{F}_p \vee \Sigma^{n+1}\mathbb{F}_p) \cong \mathrm{Der}_{\mathbb{Z}}^{n+1}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{HH}_{\mathbb{Z}}^{n+2}(\mathbb{F}_p, \mathbb{F}_p)$$

where for the last isomorphism we need $n \notin \{-1, -2\}$. Recall furthermore that $\mathrm{HH}^*(\mathbb{F}_p, \mathbb{F}_p) \cong H_{-*}[\mathrm{Hom}_{\mathbb{F}_p\otimes\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p)]$, and remember that $\mathbb{F}_p \otimes \mathbb{F}_p$ really means $\mathbb{F}_p \otimes_{\mathbb{Z}}^L \mathbb{F}_p$ here.

The dga $\mathbb{F}_p \otimes^L \mathbb{F}_p$ is $\Lambda = \Lambda_{\mathbb{F}_p}(e_1)$ (an exterior algebra with zero differential). Of course $\mathrm{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p)$ has homology algebra equal to a polynomial algebra on a class of degree -2 (or $+2$ if cohomological grading is used). That is, $\mathrm{HH}^*(\mathbb{F}_p,\mathbb{F}_p)\cong\mathbb{F}_p[\sigma_2]$.

The conclusion is that when $n\geq 1$ is odd, there is only one homotopy class in $\mathrm{Ho}\,(\mathbb{Z}-\mathcal{DGA}_{/\mathbb{F}_p})(\mathbb{F}_p,\mathbb{F}_p\vee\Sigma^{n+1}\mathbb{F}_p)$ (the trivial one), and hence only one quasi-isomorphism type for dgas whose homology is $\Lambda_{\mathbb{F}_p}(g_n)$ (given by this graded algebra with zero differential). When $n\geq 2$ is even we have

$$\mathrm{Ho}\,(\mathbb{Z}-\mathcal{DGA}_{/\mathbb{F}_p})(\mathbb{F}_p,\mathbb{F}_p\vee\Sigma^{n+1}\mathbb{F}_p)\cong\mathbb{Z}/p,$$

and quotienting by $\mathrm{Aut}(\mathbb{F}_p)=(\mathbb{F}_p)^*$ gives exactly two orbits. So in this case there are exactly two such dgas: the trivial square-zero extension (exterior algebra) and an 'exotic' one.

For example, when $n=2$ the non-trivial dga is $\mathbb{Z}[e_1;de=p]/(e^3,pe^2)$. When $n=4$ it is $\mathbb{Z}\langle e_1,f_3;de=p,df=e^2\rangle/(e^4,e^2f,efe,fe^2,,fef,f^2,p(ef+fe))$. Here the subscripts adorning the variables indicate their degrees, e.g. f_3 is an element of degree 3.

Example 3.16. Suppose that F is a field, and we want to classify all F-dgas whose homology is $\Lambda_F(g_n)$. Everything proceeds as above, and we find ourselves needing to compute $\mathrm{HH}^*_F(F,F)$. But this is trivial except when $*=0$, and so there is only one quasi-isomorphism type for the dgas in question—namely, the trivial one. The previous example is more complicated because the ground ring is $\mathbb{Z}$.

4. k-invariants for ring spectra

The material developed for dgas in the last section is developed for ring spectra in [DS3]. One has Postnikov towers of ring spectra, and the k-invariants measuring the extensions at each level now live in groups called *topological Hochschild cohomology*. We use these tools to classify ring spectra whose homotopy is $\Lambda_{\mathbb{F}_p}(g_n)$. Another reference for some of this background material is [L, Sections 2 and 8.1].

Fix a commutative ring spectrum R.

Definition 4.1. *If T is an R-algebra and $n\geq 0$, an **nth Postnikov section** of T is an R-algebra P_nT together with a map of R-algebras $T\to P_nT$ such that*

(i) $\pi_i(P_nT)=0$ for $i>n$, and
(ii) $\pi_i(T)\to\pi_i(P_nT)$ is an isomorphism for $i\leq n$.

Postnikov sections can be constructed for connective R-algebras just as they were for dgas. See [DS3, Section 2] for details. A connective R-algebra T has k-invariants lying in the set of homotopy classes $\mathrm{Ho}\,(R\text{–}\mathcal{A}lg_{/(P_nT)})(P_nT,P_nT\vee\Sigma^{n+2}H(\pi_{n+1}T))$, giving homotopy fiber sequences

$$P_{n+1}T\to P_nT\to P_nT\vee\Sigma^{n+2}H(\pi_{n+1}T).$$

This is where things diverge somewhat from what we did for dgas. For dgas we were able to produce the k-invariants in terms of very explicit formulas, defined on the elements of the dga. One cannot use this construction for ring spectra. Instead, one has to use a more categorical approach. This is developed in [DS3] and the analog of Corollary 3.10 is proven in [DS3, 8.1].

If M is a $T\wedge_R T^{op}$-module, one defines $\mathbf{TDer}_R(T,M)$ and $\mathbf{THH}_R(T,M)$ just as in the previous section.

Example 4.2. Let us investigate all ring spectra (i.e., S-algebras) T whose homotopy algebra is $\pi_* T \cong \Lambda_{\mathbb{F}_p}(g_n)$ $(n \geq 1)$. Just as for dgas, we have the single homotopy fiber sequence $T \to H\mathbb{F}_p \to H\mathbb{F}_p \vee \Sigma^{n+1} H\mathbb{F}_p$ in $\mathrm{Ho}\,(S - \mathcal{A}lg_{/H\mathbb{F}_p})$. The possibilities for the k-invariant are contained in the set $\mathrm{TDer}^{n+1}(H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathrm{THH}^{n+2}(H\mathbb{F}_p, H\mathbb{F}_p)$. These THH-groups have been calculated by Bökstedt [B], but see [HM, 5.2] for a published summary (those references deal with *THH homology*, but one can get the cohomology groups by dualization). We have $\mathrm{THH}^*(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma[\alpha_2]$, a divided polynomial algebra on a class of degree 2. As another source for this computation, including the ring structure, we refer to [FLS, 7.3].

It follows from the computation that when $n \geq 1$ is odd there is only one ring spectrum with the given homotopy algebra, namely $H(\Lambda_{\mathbb{F}_p}(g_n))$. When $n \geq 2$ is even, there are exactly two such ring spectra.

Remark 4.3. In the above example, we'd like to call special attention to the case where $n = 2p - 2$ for p a prime. In this case we can say precisely what the two homotopy types of ring spectra are. One of them, of course, is the trivial example $H(\Lambda_{\mathbb{F}_p}(g_{2p-2}))$. The other is the first nontrivial Postnikov section of connective Morava K-theory, $P_{2p-2}k(1)$ (see [A] or [G] for the ring structure on $k(1)$ when $p = 2$). These two ring spectra are obviously not weakly equivalent, since their underlying spectra are not even weakly equivalent (the latter is not an Eilenberg-MacLane spectrum). So they must represent the two homotopy types.

4.4. **Comparing** HH **and** THH. Suppose $Q \to R$ is a map of commutative ring spectra, and T is an R-algebra. There is a natural map $T \wedge_Q T^{op} \to T \wedge_R T^{op}$, and as a result a natural map

$$\mathbf{THH}_R(T, M) = \mathrm{Hom}_{T \wedge_R T^{op}}(T, M) \to \mathrm{Hom}_{T \wedge_Q T^{op}}(T, M) = \mathbf{THH}_Q(T, M).$$

(Note that the smash means 'derived smash' and the hom means 'derived hom', as will always be the case in this paper). In particular we may apply this when $Q \to R$ is the map $S \to H\mathbb{Z}$. If T is an $H\mathbb{Z}$-algebra we obtain a map $\mathbf{THH}_{H\mathbb{Z}}(T, M) \to \mathbf{THH}_S(T, M)$. By the equivalence of $H\mathbb{Z}$-algebras with dgas, $\mathbf{THH}_{H\mathbb{Z}}$ is just another name for $\mathbf{HH}_{\mathbb{Z}}$. So if T is a dga and M is a T-bimodule, we are saying that there are natural ring maps

$$\mathrm{HH}^*_{\mathbb{Z}}(T, M) \to \mathrm{THH}^*_S(HT, HM).$$

When M is concentrated entirely in degree 0 (as in all our application), one can show that for $n \geq 1$ the following square commutes:

$$\begin{array}{ccc} \mathrm{Ho}\,(H\mathbb{Z} - \mathcal{A}lg_{/HT})(HT, HT \vee \Sigma^{n+1} HM) & \xrightarrow{\cong} & \mathrm{THH}^{n+2}_{H\mathbb{Z}}(T, M) \\ \downarrow & & \downarrow \\ \mathrm{Ho}\,(S - \mathcal{A}lg_{/HT})(HT, HT \vee \Sigma^{n+1} HM) & \xrightarrow{\cong} & \mathrm{THH}^{n+2}_S(HT, HM). \end{array}$$

Then, using the identification of $H\mathbb{Z}$-algebras with dgas, the top horizontal map can be identified with

$$\mathrm{Ho}\,(\mathbb{Z} - \mathcal{DGA}_{/T})(T, T \vee \Sigma^{n+1} M) \to \mathrm{HH}^{n+2}_{\mathbb{Z}}(T, M).$$

5. Examples of topological equivalence

In this section we present several examples of dgas which are topologically equivalent but not quasi-isomorphic. Since the homotopy theory of dgas is Quillen equivalent to that of $H\mathbb{Z}$-algebras, this is the same as giving two non-equivalent $H\mathbb{Z}$-algebra structures on the same underlying ring spectrum. Our examples are:

(a) The dgas $\mathbb{Z}[e_1; de = 2]/(e^4)$ and $\Lambda_{\mathbb{F}_2}(g_2)$.
(b) The two distinct quasi-isomorphism types of dgas whose homology is the exterior algebra $\Lambda_{\mathbb{F}_p}(g_{2p-2})$, provided by Example 3.15. (This gives the example from (a) when $p = 2$).
(c) The ring spectrum $H\mathbb{Z}\wedge H\mathbb{Z}/2$, with the two structures of $H\mathbb{Z}$-algebra provided by the two maps $H\mathbb{Z} = H\mathbb{Z}\wedge S \to H\mathbb{Z}\wedge H\mathbb{Z}/2$ and $H\mathbb{Z} = S\wedge H\mathbb{Z} \to H\mathbb{Z}\wedge H\mathbb{Z}/2$.
(d) The ring spectrum $H\mathbb{Z}\wedge_{bo}H\mathbb{Z}/2$, with the two structures of $H\mathbb{Z}$-algebra coming from the left and the right as in (c).

For parts (a) and (b), we must prove that these dgas are topologically equivalent—we do this by using the comparison map from Hochschild cohomology to topological Hochschild cohomology. For parts (c) and (d), we must prove that the associated dgas are not quasi-isomorphic—we do this by calculating the derived tensor with $\mathbb{Z}/2$, and finding that we get different homology rings.

The examples in (a), (c), and (d) are related. Specifically, the two dgas in (a) are the second Postnikov sections of the dgas in (d) (or in (c)).

5.1. **Examples using** HH **and** THH**.** In this section we will mainly be working with $\mathbb{Z}$-dgas and with S-algebras. The symbols HH^* and THH^* will always indicate $\mathrm{HH}^*_{\mathbb{Z}}$ and THH^*_S, unless otherwise noted.

We begin with the following simple result:

Proposition 5.2. *Let C be a dga, and let P_nC be an nth Postnikov section. Then $HC \to H(P_nC)$ is an nth Postnikov section for the ring spectrum HC.*

Proof. Immediate. □

Suppose C is a non-negative $\mathbb{Z}$-dga with $P_nC \simeq C$, and let X be a Postnikov $(n + 1)$-extension of C. Write $M = H_{n+1}X$. We have a homotopy fiber sequence $X \to C \to C \vee \Sigma^{n+2}M$ in $\mathrm{Ho}\,(\mathbb{Z} - DGA_{/C})$. Since $H(-)$ preserves homotopy limits, applying it yields a homotopy fiber sequence $HX \to HC \to HC \vee \Sigma^{n+2}HM$ in $\mathrm{Ho}\,(\mathcal{R}ing\mathcal{S}p/HC)$. So HX is a Postnikov $(n + 1)$-extension of HC. The k-invariant for X lies in $\mathrm{HH}^{n+3}(C, M)$, whereas the k-invariant for HX lies in $\mathrm{THH}^{n+3}(HC, HM)$. The latter is the image of the former under the natural map $\mathrm{HH}^{n+3}(C, M) \to \mathrm{THH}^{n+3}(HC, HM)$.

We can now give our first example of two dgas which are topologically equivalent but not quasi-isomorphic. The example will be based on our knowledge of the map $\mathrm{HH}^*(\mathbb{F}_p, \mathbb{F}_p) \to \mathrm{THH}^*(\mathbb{F}_p, \mathbb{F}_p)$. The domain is $\mathbb{F}_p[\sigma_2]$ (as calculated in 3.15), whereas the codomain is the divided power algebra $\Gamma_{\mathbb{F}_p}[\alpha_2]$ (cf. [FLS, 7.3]). Recall that in characteristic p one has

$$\Gamma_{\mathbb{F}_p}[\alpha_2] \cong \mathbb{F}_p[e_2, e_{2p}, e_{2p^2}, \ldots]/(e_2^p, e_{2p}^p, \ldots)$$

It is easy to see that the map $\mathrm{HH}^2(\mathbb{F}_p, \mathbb{F}_p) \to \mathrm{THH}^2(\mathbb{F}_p, \mathbb{F}_p)$ is an isomorphism—the k-invariants in $\mathrm{HH}^2(\mathbb{F}_p, \mathbb{F}_p)$ classify the two dgas $\mathbb{Z}/p^2$ and $\mathbb{Z}/p[\epsilon]/\epsilon^2$, and the k-invariants in $\mathrm{THH}^2(\mathbb{F}_p, \mathbb{F}_p)$ classify the ring spectra $H\mathbb{Z}/p^2$ and $H(\mathbb{Z}/p[\epsilon]/\epsilon^2)$.

So by choosing our generators appropriately we can assume σ is sent to α. We therefore have that $\sigma^p \in \mathrm{HH}^{2p}(\mathbb{F}_p, \mathbb{F}_p)$ maps to zero in $\mathrm{THH}^{2p}(\mathbb{F}_p, \mathbb{F}_p)$, using the ring structure.

Let C be the $\mathbb{Z}$-dga whose homology is $\Lambda_{\mathbb{F}_p}(g_{2p-2})$ and whose nontrivial k-invariant is σ^p. Let D be the dga $\Lambda_{\mathbb{F}_p}(g_{2p-2})$ with zero differential. We know C is not quasi-isomorphic to D, as they have different k-invariants in HH^*. However, the k-invariants for the ring spectra HC and HD are the same (and are equal to the zero element of $\mathrm{THH}^{2p}(\mathbb{F}_p, \mathbb{F}_p)$). So $HC \simeq HD$, that is to say C and D are topologically equivalent.

To be even more concrete, take $p = 2$. Then $C \simeq \mathbb{Z}[e; de = 2]/(e^4)$ and $D = \Lambda_{\mathbb{F}_2}(g_2)$. We have shown that these are topologically equivalent, but not quasi-isomorphic.

It's worth observing that as p increases the dgas produced by this example become more complicated; to construct them explicitly one is required to go further and further out in the resolution of $\mathbb{F}_p$ as a $\mathbb{Z}$-dga.

Remark 5.3. Note that $\mathrm{HH}^*(\mathbb{F}_p, \mathbb{F}_p)$ and $\mathrm{THH}^*(\mathbb{F}_p, \mathbb{F}_p)$ are isomorphic as abstract groups (equal to zero in odd dimensions, $\mathbb{F}_p$ in even dimensions). It is somewhat of a surprise that the map between them is not an isomorphism. The reader should take note that even without knowledge of the ring structures on HH^* and THH^*, one can still see that σ^p must map to zero. The map $\mathrm{HH}^{2p} \to \mathrm{THH}^{2p}$ has the form $\mathbb{Z}/p \to \mathbb{Z}/p$, and the k-invariant for the ring spectrum $P_{2p-2}k(1)$ is certainly a non-trivial element in the latter group (here $k(1)$ is the first Morava K-theory spectrum; see [A] or [G] for the ring structure when $p = 2$). But this ring spectrum cannot possibly be $H(-)$ of any dga, since the underlying spectrum is not Eilenberg-MacLane. So the map $\mathbb{Z}/p \to \mathbb{Z}/p$ cannot be surjective—and hence not injective, either.

5.4. **Examples using $H\mathbb{Z}$-algebra structures.** The following examples give another approach to producing topologically equivalent dgas. For these examples, note that if A is a $\mathbb{Z}$-dga then by the **(derived) mod 2 homology ring** of A we mean the ring $H_*(A \otimes^L \mathbb{Z}/2)$.

Example 5.5. The problem is to give two dgas which are topologically equivalent but not quasi-isomorphic. This is equivalent—via the identification of dgas and H$\mathbb{Z}$-algebras from [S1]—to giving two $H\mathbb{Z}$-algebras which are weakly equivalent as ring spectra, but not as $H\mathbb{Z}$-algebras. Consider the ring spectrum $H\mathbb{Z} \wedge_S H\mathbb{Z}/2$, which has two obvious $H\mathbb{Z}$-algebra structures (from the left and right sides of the smash). Let HC and HD denote these two different $H\mathbb{Z}$-algebra structures, coming from the left and from the right respectively. We claim that HC and HD are not weakly equivalent as $H\mathbb{Z}$-algebras—although obviously they are the same ring spectrum. This will give another example of topological equivalence.

To see that HC and HD are distinct $H\mathbb{Z}$-algebras, we can give the following argument (it is inspired by one shown to us by Bill Dwyer). If they *were* equivalent, then one would have an equivalence of ring spectra $HC \wedge_{H\mathbb{Z}} E \simeq HD \wedge_{H\mathbb{Z}} E$ for any $H\mathbb{Z}$-algebra E, and therefore a resulting isomorphism of rings $\pi_*(HC \wedge_{H\mathbb{Z}} E) \cong \pi_*(HD \wedge_{H\mathbb{Z}} E)$. Write $H = H\mathbb{Z}$ and let's choose $E = H\mathbb{Z}/2$. So we will be computing the (derived) mod 2 homology rings of the associated dgas C and D, since $\pi_*(HC \wedge_{H\mathbb{Z}} E) \cong H_*(C \otimes^L \mathbb{Z}/2)$.

For HC we have

$$HC \wedge_H E = E \wedge_H HC = E \wedge_H (H \wedge E) = E \wedge E.$$

So $\pi_*(HC \wedge_H E) = \pi_*(E \wedge E)$ and we have that the mod 2 homology ring of C is the dual Steenrod algebra.

For HD, however, the situation is different. The structure map from $H\mathbb{Z}$ to HD factors through $H\mathbb{Z}/2$, and so D is an $\mathbb{F}_2$-dga. One readily checks that for any $\mathbb{F}_2$-dga the mod 2 homology has an element of degree 1 whose square is zero (in fact if X is an $\mathbb{F}_2$-dga then $H_*(X \otimes^L \mathbb{Z}/2) \cong H_*(X) \otimes_{\mathbb{F}_2} \Lambda_{\mathbb{F}_2}(e_1)$. We find, therefore, that C and D have different mod 2 homology rings—since the dual Steenrod algebra does not have an element in degree 1 squaring to zero. So C and D cannot be quasi-isomorphic.

Example 5.6. In the previous example, the $H\mathbb{Z}$-algebras HC and HD are quite big—having nonvanishing homotopy groups in infinitely many degrees. One can obtain a smaller example by letting HC and HD be $H\mathbb{Z} \wedge_{bo} H\mathbb{Z}/2$, with the $H\mathbb{Z}$-algebra structure coming from the left and right, respectively. One applies the same analysis as before to see that C and D—the associated dgas—have different mod 2 homology rings. One only needs to know that $\pi_*(HC \wedge_H E) \cong \pi_*(E \wedge_{bo} E) \cong A(1)_* = A_*/(\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, ...)$, where A_* is the dual Steenrod algebra and $\bar{\xi}_i = c(\xi_i)$ where c is the anti-automorphism. In particular, this ring does not have an element of degree 1 which squares to zero.

Note that the homotopy rings of C and D are just $\Lambda_{\mathbb{F}_2}(f_2, g_3)$. To see this, use our knowledge of $\pi_*(E \wedge_{bo} E)$ together with an analysis of the cofiber sequence of spectra

$$H \wedge_{bo} E \xrightarrow{2} H \wedge_{bo} E \to E \wedge_{bo} E$$

(note that the multiplication by 2 map is null). This shows that the homotopy of C is $\mathbb{Z}/2$ in degrees 0, 2, 3, and 5, and that the multiplication is commutative—but it does not immediately identify the class in degree 5 as a product. However, the mod 2 homology of C is $A(1)_*$ (as $E \wedge_H C \simeq E \wedge_{bo} E$) and this will only happen when the degree 5 class is the product of the classes in degrees 2 and 3.

In fact one can determine the dgas C and D explicitly: D is $\Lambda_{\mathbb{F}_2}(g_2, h_3)$ with zero differential, and C is the dga $\mathbb{Z}\langle e_1, h_3; de = 2, dh = 0\rangle/(e^4, h^2, eh + he)$. For D this follows because it is an $\mathbb{F}_2$-algebra, and an analysis of the possible k-invariants in the Postnikov tower shows there is only one $\mathbb{F}_2$-dga with the given homology ring. For C, when analyzing the Postnikov tower one finds there are two possibilities for P_2C (the two dgas given near the end of Section 5.1). The only one which gives the correct mod 2 homology of C is $\mathbb{Z}[e; de = 2]/(e^4)$. After this stage, at each level of the Postnikov tower there is only one possible k-invariant consistent with the given homology ring.

Finally, note that we cannot get a similar example by using the left and right $H\mathbb{Z}$-algebra structures on $H\mathbb{Z} \wedge_{bu} H\mathbb{Z}/2$. Unlike the previous examples, these actually give weakly equivalent $H\mathbb{Z}$-algebras. To see this, apply $(-) \wedge_{bu} H\mathbb{Z}/2$ to the cofiber sequence $\Sigma^2 bu \to bu \to H\mathbb{Z}$; this shows that the homotopy ring of $H\mathbb{Z} \wedge_{bu} H\mathbb{Z}/2$ is $\Lambda_{\mathbb{F}_2}(e_3)$. We showed in Example 3.15 that there is only one homotopy type for dgas with this homology ring.

5.7. **Topological equivalence over fields.** For the above examples of nontrivial topological equivalence, it has been important in each case that we were dealing with dgas over $\mathbb{Z}$ whose zeroth homology is $\mathbb{Z}/p$. In each example one of the dgas involved a class in degree 1 whose differential is p times the unit. This leads to the following two questions:

Question 1: Do there exists two dgas over $\mathbb{Z}$ which have torsion-free homology, and which are topologically equivalent but not quasi-isomorphic?

Question 2: Let F be a field. Do there exist two F-dgas which are topologically equivalent but not quasi-isomorphic (as F-dgas)?

So far we have been unable to answer these questions except in one simple case. We can show that if two $\mathbb{Q}$-dgas are topologically equivalent then they must actually be quasi-isomorphic:

Proof of Proposition 1.7. The key here is that for any ring spectrum R with rational homotopy, the map $\eta\colon R = R \wedge_S S \to R \wedge_S H\mathbb{Q}$ is a weak equivalence. One way to see this is to note that $\mathbb{Q}$ is flat over π_*^s, as it is the localization of $\pi_* S$ at the set of nonzero integers. So the spectral sequence calculating the homotopy of $X \wedge_S H\mathbb{Q}$ from [EKMM, IV.4.2] collapses, showing that η induces isomorphisms on homotopy.

If A is a $\mathbb{Q}$-dga then the above shows that $\eta_A\colon HA \to HA \wedge_S H\mathbb{Q}$ is a weak equivalence of ring spectra.

Assume C and D are two topologically equivalent $\mathbb{Q}$-dgas. Then HC and HD are weakly equivalent as S-algebras. It follows that $HC \wedge_S H\mathbb{Q}$ and $HD \wedge_S H\mathbb{Q}$ are equivalent as $H\mathbb{Q}$-algebras. If η_C and η_D were $H\mathbb{Q}$-algebra maps we could conclude that HC and HD were equivalent as $H\mathbb{Q}$-algebras, and hence that C and D are quasi-isomorphic $\mathbb{Q}$-dgas. Unfortunately, the claim that η_C and η_D are $H\mathbb{Q}$-algebra maps is far from clear. Instead, we will use the map $\psi_C\colon HC \wedge_S H\mathbb{Q} \to HC \wedge_{H\mathbb{Q}} H\mathbb{Q} \cong HC$. This is a map of $H\mathbb{Q}$-algebras. Note that $\psi_C \eta_C$ is the identity, and so ψ_C is also a weak equivalence.

Using ψ_C and ψ_D we obtain a zig-zag of weak equivalences of $H\mathbb{Q}$-algebras $HC \xleftarrow{\sim} HC \wedge_S H\mathbb{Q} \simeq HD \wedge_S H\mathbb{Q} \xrightarrow{\sim} HD$. So C and D are quasi-isomorphic as $\mathbb{Q}$-dgas. □

6. Homotopy endomorphism spectra and dgas

The next main goal is the proof of our Tilting Theorem (7.2). The portion of the proof requiring the most technical difficulty involves keeping track of information preserved by a zig-zag of Quillen equivalences. The machinery needed to handle this is developed in [D2] and [DS2]. The present section summarizes what we need.

A model category is called **combinatorial** if it is cofibrantly-generated and the underlying category is locally presentable. See [D1] for more information. The categories of modules over a dga and modules over a symmetric ring spectrum are both combinatorial model categories.

If $\mathcal{M}$ is a combinatorial, stable model category then [D2] explains how to associate to any object $X \in \mathcal{M}$ a **homotopy endomorphism ring spectrum** $\mathrm{hEnd}(X)$. This should really be regarded as an isomorphism class in $\mathrm{Ho}\,(S - \mathcal{A}lg)$, but we will usually act as if a specific representative has been chosen.

Proposition 6.1. [D2, Thm. 1.4] *Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial, stable model categories. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are connected by a zig-zag of Quillen equivalences (in which no assumptions are placed on the intermediate model categories in the zig-zag). Suppose that $X \in \mathcal{M}$ and $Y \in \mathcal{N}$ correspond under the derived equivalence* *of homotopy categories. Then* hEnd(X) *and* hEnd(Y) *are weakly equivalent ring spectra.*

Recall that a category is said to be *additive* if the Hom-sets have natural structures of abelian groups, the composition is bilinear, and the category has finite coproducts. See [ML, Section VIII.2]. Such a category is necessarily pointed: the empty coproduct is an initial object, which is also a terminal object by [ML, VIII.2.1].

By an **additive model category** we mean a model category whose underlying category is additive and where the additive structure interacts well with the 'higher homotopies'. See [DS2, Section 2] for a precise definition, which involves the use of cosimplicial resolutions. If R is a dga, the model category Mod- R of differential graded R-modules is one example of an additive model category; this example is discussed in more detail at the beginning of the next section.

Note that if $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ is a Quillen pair where $\mathcal{M}$ and $\mathcal{N}$ are additive, then both L and R are additive functors—this is because they preserve direct sums (equivalently, direct products) since L preserves colimits and R preserves limits.

If $\mathcal{M}$ and $\mathcal{N}$ are two additive model categories, we say they are **additively Quillen equivalent** if they can be connected by a zig-zag of Quillen equivalences where all the intermediate categories are additive. As remarked in the introduction, it is possible for two additive model categories to be Quillen equivalent but not additively Quillen equivalent. We'll give an example in Section 8 below.

6.2. *Ch* **enrichments.** Recall that *Ch* denotes the model category of chain complexes of abelian groups, with its projective model structure. A *Ch*-model category is a model category with compatible tensors, cotensors, and enrichments over *Ch*; see [Ho1, 4.2.18] or [D2, Appendix A]. For X, Y in $\mathcal{M}$, we denote the function object in *Ch* by $\underline{\mathcal{M}}_{\mathrm{Ch}}(X,Y)$.

Note that a pointed *Ch*-model category is automatically an additive and stable model category. The additivity of the underlying category, for instance, follows from the isomorphisms $\mathcal{M}(X,Y) \cong \mathcal{M}(X \otimes \mathbb{Z}, Y) \cong Ch(\mathbb{Z}, \underline{\mathcal{M}}_{\mathrm{Ch}}(X,Y))$ and the fact that the last object has a natural structure of abelian group. This additive structure is also compatible with the higher homotopies; see [DS2, 2.9]. The stability follows directly from the stability of *Ch*; cf. [SS2, 3.5.2] or [GS, 3.2].

We will need the following result, which is proven in [DS2], connecting *Ch*-enrichments to homotopy endomorphism spectra:

Proposition 6.3. *Let $\mathcal{M}$ be a combinatorial Ch-model category such that $\mathcal{M}$ has a generating set of compact objects (cf. Definition 7.1). Let $X \in \mathcal{M}$ be a cofibrant-fibrant object. Then*

(a) hEnd(X) *is weakly equivalent to the Eilenberg-MacLane ring spectrum associated to the dga $\underline{\mathcal{M}}_{\mathrm{Ch}}(X,X)$.*
(b) Suppose $\mathcal{N}$ is another combinatorial, Ch-model category with a generating set of compact objects, and that $\mathcal{M}$ and $\mathcal{N}$ are connected by a zig-zag of additive Quillen equivalences. Let Y be a cofibrant-fibrant object corresponding to X under the derived equivalence of homotopy categories. Then the dgas $\underline{\mathcal{M}}_{\mathrm{Ch}}(X,X)$ and $\underline{\mathcal{N}}_{\mathrm{Ch}}(Y,Y)$ are quasi-isomorphic.

Part (a) of the above result follows directly from [DS2, 1.4, 1.6]. Likewise, part (b) follows directly from [DS2, 1.3, 1.6].

7. Tilting theory

This section addresses the following question: given two dgas C and D, when are the model categories of (differential graded) C-modules and D-modules Quillen equivalent? We give a complete answer in terms of topological tilting theory, making use of topological equivalence. There is also the associated question of when the two module categories are *additively* Quillen equivalent, which can be answered with a completely algebraic version of tilting theory.

If C is a dga, let Mod-C be the category of (right) differential graded C-modules. This has a model structure lifted from Ch, in which a map is a fibration or weak equivalence if and only if the underlying map of chain complexes is so; see [SS1, 4.1(1)]. We let $\mathcal{D}(C)$ denote the homotopy category of Mod-C, and call this the **derived category** of C.

The category of C-modules is enriched over Ch: for X, Y in Mod-C, let $\mathrm{Hom}_C(X, Y)$ be the chain complex which in degree n consists of C-module homomorphisms of degree n on the underlying graded objects (ignoring the differential). The differential on $\mathrm{Hom}_C(X, Y)$ is then defined so that the chain maps are the cycles. See [Ho1, 4.2.13].

Definition 7.1. *Let $\mathcal{T}$ be a triangulated category with infinite coproducts.*

*(a) An object $P \in \mathcal{T}$ is called **compact** if $\oplus_\alpha \mathcal{T}(P, X_\alpha) \to \mathcal{T}(P, \oplus_\alpha X_\alpha)$ is an isomorphism for every set of objects $\{X_\alpha\}$;*

*(b) A set of objects $S \subseteq \mathcal{T}$ is a **generating set** if the only full triangulated subcategory of $\mathcal{T}$ which contains S and is closed under arbitrary coproducts is $\mathcal{T}$ itself. If S is a singleton set $\{P\}$ we say that P is a **generator**.*

An object X in a model category $\mathcal{M}$ is called compact if it is a compact object of the associated homotopy category. Likewise for the notion of a generating set.

We can now state the main result of this section:

Theorem 7.2 (Tilting Theorem). *Let C and D be two dgas.*

(a) The model categories of C-modules and D-modules are Quillen equivalent if and only if there is a cofibrant and fibrant representative P of a compact generator in $\mathcal{D}(C)$ such that $\mathrm{Hom}_C(P, P)$ is topologically equivalent to D.

(b) The model categories of C-modules and D-modules are additively *Quillen equivalent if and only if there is a cofibrant and fibrant representative P of a compact generator in $\mathcal{D}(C)$ such that $\mathrm{Hom}_C(P, P)$ is quasi-isomorphic to D.*

Before proving this theorem, we need to recall a few results on ring spectra and their module categories. If R is a ring spectrum, we again let Mod-R denote the category of right R-modules equipped with the model structure of [SS1, 4.1(1)]. This model category is enriched over symmetric spectra: for $X, Y \in$ Mod-R we let $\mathrm{Hom}_R(X, Y)$ denote the symmetric spectrum mapping object.

Proposition 7.3. *(a)* [SS2, 4.1.2] *Let R be a ring spectrum, and let P be a cofibrant and fibrant representative of a compact generator of the homotopy category of R-modules. Then there is a Quillen equivalence* Mod-$\mathrm{Hom}_R(P, P) \to$ Mod-R.

(b) [GS, 3.4] *Let C be a dga, and let P be a cofibrant and fibrant representative of a compact generator of the homotopy category of C-modules. Then there is a Quillen equivalence* Mod-$\mathrm{Hom}_C(P, P) \to$ Mod-C.

In [GS, 3.4] this is actually proved over $\mathbb{Q}$, but the same proofs work over $\mathbb{Z}$.

Proposition 7.4. *(a)* [HSS, 5.4.5] *If $R \to T$ is a weak equivalence of ring spectra, then there is an induced Quillen equivalence* Mod-$R \to$ Mod-T.

(b) [SS1, 4.3] *If $C \to D$ is a quasi-isomorphism of dgas, then there is an induced Quillen equivalence* Mod-$C \to$ Mod-D.

Now we can prove the Tilting Theorem:

Proof of Theorem 7.2. For part (a), note first that if Mod-C and Mod-D are Quillen equivalent then $\mathcal{D}(C)$ and $\mathcal{D}(D)$ are triangulated equivalent. Let P be the image of D in $\mathcal{D}(C)$. Since D is a compact generator of $\mathcal{D}(D)$, its image P is a compact generator of $\mathcal{D}(C)$. But Mod-C and Mod-D are combinatorial, stable model categories, so by Proposition 6.1 we know hEnd(P) and hEnd(D) are weakly equivalent ring spectra. The two module categories are also Ch-categories, so if follows from Proposition 6.3(a) that

$$\mathrm{hEnd}(P) \simeq H(\mathrm{Hom}_C(P,P)) \qquad \text{and} \qquad \mathrm{hEnd}(D) \simeq H(\mathrm{Hom}_D(D,D)).$$

Since $\mathrm{Hom}_D(D,D)$ is isomorphic to D, we have that $\mathrm{Hom}_C(P,P)$ and D are topologically equivalent.

For the other direction, we are given that $H\,\mathrm{Hom}_C(P,P)$ and HD are weakly equivalent as ring spectra. Thus $H\,\mathrm{Hom}_C(P,P)$-modules and HD-modules are Quillen equivalent by Proposition 7.4(a). By [S1, 2.8], the model category of $\mathrm{Hom}_C(P,P)$-modules is Quillen equivalent to $H\,\mathrm{Hom}_C(P,P)$-modules and similarly for D-modules and HD-modules. So Mod-D and Mod-$\mathrm{Hom}_C(P,P)$ are Quillen equivalent. Proposition 7.3(a) then finishes the string of Quillen equivalences by showing that Mod-C and Mod-$\mathrm{Hom}_C(P,P)$ are Quillen equivalent.

Now we turn to part (b) of the theorem. If the categories of C-modules and D-modules are additively Quillen equivalent then the image of D in $\mathcal{D}(C)$ is a compact generator, just as in part (a). By Proposition 6.3(b), $\mathrm{Hom}_D(D,D)$ is quasi-isomorphic to $\mathrm{Hom}_C(P,P)$. We have already remarked that D and $\mathrm{Hom}_D(D,D)$ are isomorphic, so D is quasi-isomorphic to $\mathrm{Hom}_C(P,P)$.

For the other direction, suppose given a compact generator P in $\mathcal{D}(C)$. Proposition 7.3(b) shows that Mod-C is additively Quillen equivalent to Mod-$\mathrm{Hom}_C(P,P)$, and Proposition 7.4(b) shows that Mod-$\mathrm{Hom}_C(P,P)$ is additively Quillen equivalent to Mod-D. □

Remark 7.5. When R and S are rings, the following statements are equivalent:

(1) $\mathcal{D}(R)$ is triangulated-equivalent to $\mathcal{D}(S)$;

(2) There is a cofibrant and fibrant representative P of a compact generator in $\mathcal{D}(S)$ such that the dga $\mathrm{Hom}_S(P,P)$ is quasi-isomorphic to R;

(3) Ch_R and Ch_S are Quillen equivalent model categories.

The equivalence of (1) and (2) was established by Rickard [Ri], and the equivalence with (3) was explicitly noted in [DS1, 2.6] (this reference only discusses *pointed* Quillen equivalences, but that restriction can be removed using [D2, 5.5(b)]: if Ch_R and Ch_S are Quillen equivalent, then they are Quillen equivalent through a zig-zag of pointed model categories).

When R and S are dgas—rather than just rings—the situation changes somewhat. Theorem 7.2 establishes that the analogs of (2) and (3) are still equivalent, where in (2) "quasi-isomorphic" is replaced by "topologically equivalent".

And (3) certainly implies (1). But the implication (1) $\Rightarrow$ (2) is not true. One counterexample is $R = \mathbb{Z}\langle e_1, x_1, x^{-1}; de = p, dx = 0\rangle/(ex + xe = x^2)$ and $S = H_*(R) = \mathbb{Z}/p[x_1, x^{-1}; dx = 0]$. The verification that this is indeed a counterexample will be taken up in the paper [DS4]. The dgas R and S arise in connection with the stable module categories discussed in [Sch].

8. A model category example

In this section we give an example of two additive model categories which are Quillen equivalent but not additively Quillen equivalent.

Let C and D be the dgas $\mathbb{Z}[e_1; de = 2]/(e^4)$ and $\Lambda_{\mathbb{Z}/2}(g_2)$. We have already seen in Section 5.1 that these dgas are topologically equivalent. Therefore Mod-C and Mod-D are Quillen equivalent model categories (by Theorem 7.2(a), for instance). We claim that they are not *additively* Quillen equivalent, however. If they were, then by Theorem 7.2 there would be a compact generator P in Mod-D such that the dga $\mathrm{Hom}_D(P, P)$ is quasi-isomorphic to C. But since everything in Mod-D is a $\mathbb{Z}/2$-module, $\mathrm{Hom}_D(P, P)$ is a $\mathbb{Z}/2$-module as well. We will be done if we can show that C is not quasi-isomorphic to any dga defined over $\mathbb{Z}/2$.

Assume V is a $\mathbb{Z}/2$-dga, and C is quasi-isomorphic to V. Let $Q \to C$ be the cofibrant-replacement for C constructed as in Example 2.3. Our assumption implies that there is a weak equivalence $Q \to V$. The map $Q_0 \to V_0$ is completely determined, since $Q_0 = \mathbb{Z}$ and the unit must map to the unit. The map $Q_1 \to V_1$ must send e to an element $E \in V_1$ such that $dE = 0$ (using that $2V_0 = 0$ and $de = 2$). But $H_1(V) = H_1(C) = 0$, and so $E = dX$ for some $X \in V_2$. Note that we then have $d(EX) = -E^2$ by the Leibniz rule. However, the generator of $H_2(Q)$ is e^2, and we have just seen that the image of e^2 is zero in homology (since E^2 is a boundary). This contradicts the map $H_2(Q) \to H_2(V)$ being an isomorphism. This completes our example.

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