MATH 512 MORE NOTES ON FORCING

Some definitions:

Definition 1. *A poset $\mathbb{P}$ has the κ-chain condition if every maximal antichain $A \subset \mathbb{P}$ has size less than κ. If $\kappa = \omega_1$, this is just the countable chain condition.*

Definition 2. *A poset $\mathbb{P}$ is κ-closed if for every $\tau < \kappa$ and every decreasing sequence $\langle p_\alpha \mid \alpha < \tau \rangle$ (i.e. $\alpha < \beta \to p_\beta \leq p_\alpha$), there is a condition $p \in \mathbb{P}$, such that $p \leq p_\alpha$ for all α.*

Preservation of cardinals:

Let $\mathbb{P}$ be a poset. If λ is a cardinal in V, we say $\mathbb{P}$ *preserves* λ if for any generic filter G, $V[G] \models$ "λ is a cardinal." If on the other hand for any generic filter G, $V[G] \models$ "λ is not a cardinal", then $\mathbb{P}$ *collapses* λ.

Fact 3. *Let $\mathbb{P}$ be a poset and κ be a regular uncountable cardinal.*

1. *If $\mathbb{P}$ has the κ-chain condition, then $\mathbb{P}$ preserves all cardinals greater than or equal to κ.*
2. *If $\mathbb{P}$ is κ-closed, then $\mathbb{P}$ preserves all cardinals less than or equal to κ.*

Projections

Definition 4. *$\pi : \mathbb{P} \to \mathbb{Q}$ is a projection if:*

1. *if $p' \leq p$, then $\pi(p') \leq \pi(p)$.*
2. *for every $p \in \mathbb{P}$, if $q \leq \pi(p)$, then there is $p' \leq p$, such that $\pi(p') \leq q$.*

We say that $\mathbb{P}$ projects to $\mathbb{Q}$, if there is a projection as above.

Lemma 5. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, and G is $\mathbb{P}$-generic. Then $H := \{q \in \mathbb{Q} \mid \exists p \in G, \pi(p) \leq q\}$ is $\mathbb{Q}$-generic.*

Proof. H is upward closed by definition. Also if $q_1, q_2 \in H$, let $p_1, p_2 \in G$ be such that $\pi(p_1) \leq q_1, \pi(p_2) \leq q_2$. Let $p \in G$ be stronger than both p_1, p_2. Then $\pi(p) \in H$ is stronger than both q_1, q_2. So H is a filter.

For genericity, suppose that D is a dense subset of $\mathbb{Q}$. Define $\bar{D} := \{p \mid (\exists q)(\pi(p) \leq q, q \in D\}$. We want to show that this is dense. Let $p \in \mathbb{P}$, and let $q \leq \pi(p)$ be such that $q \in D$ (here we use density of D). Then, since π is a projection, let $p' \leq p$ be such that $\pi(p') \leq q$. Then $p' \in \bar{D}$. So, $\bar{D}$ is a dense subset of $\mathbb{P}$.

Now let $p \in \bar{D} \cap G$. That means there is some $q \in D$ with $\pi(p) \leq q$. Then $q \in D \cap H$.

$\square$

Products and iterations

Definition 6. *For posets $\mathbb{P}, \mathbb{Q}$ in V, the product is $\mathbb{P} \times \mathbb{Q} := \{(p, q) \mid p \in \mathbb{P}, q \in \mathbb{Q}\}$, and $(p', q') \leq (p, q)$ iff $p' \leq p$ and $q' \leq q$.*

Lemma 7. *(The product lemma) Let K be $\mathbb{P} \times \mathbb{Q}$-generic. Let $G = \{p \mid (\exists q)(p, q) \in K\}$ and $H = \{q \mid (\exists p)(p, q) \in K\}$. Then G is $\mathbb{P}$-generic over V, and H is $\mathbb{Q}$-generic over $V[G]$.*

Conversely, let G be $\mathbb{P}$-generic over V, and H be $\mathbb{Q}$-generic over $V[G]$. Then $K := \{(p, q) \mid p \in G, q \in H\}$ is $\mathbb{P} \times \mathbb{Q}$-generic over V.

In both cases we have that $V[K] = V[G][H] = V[H][G]$. Also, since $\mathbb{P} \times \mathbb{Q}$ projects to both $\mathbb{P}$ and $\mathbb{Q}$, we get $V \subset V[G] \subset V[K]$ and $V \subset V[H] \subset V[K]$.

Definition 8. *In V, let $\mathbb{P}$ be a poset, and suppose that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset. Define the two step iteration, $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, \dot{q}) \mid p \in \mathbb{P}, 1_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$, with the following ordering:*

$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ iff:

1. *$p_1 \leq p_2$,*
2. *$p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \leq \dot{q}_2$.*

Lemma 9. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a two step iteration. Suppose that K is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V. Then $G = \{p \mid (\exists \dot{q})(p, \dot{q}) \in K\}$ is $\mathbb{P}$-generic over V and $H := \{\dot{q}_G \mid (\exists p \in G)(p, \dot{q}) \in K\}$ is $\dot{\mathbb{Q}}_G$-generic over $V[G]$.*

Conversely, if G is $\mathbb{P}$-generic over V and H is $\dot{\mathbb{Q}}_G$-generic over $V[G]$, then $K = \{(p, \dot{q}) \mid p \in G, \dot{q}_G \in H\}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.

$\mathbb{P} * \dot{\mathbb{Q}}$ projects to $\mathbb{P}$, and denoting the generics as above we have that, $V \subset V[G] \subset V[G][H] = V[K]$.

Some examples:

1. The Cohen poset to add a real, $Add(\omega, 1) := \{f : \omega \to \{0, 1\} \mid |\operatorname{dom}(f)| < \omega\}$ i.e. the set of all finite partial functions from ω to $\{0, 1\}$. The order is reverse inclusion: $f \leq g$ iff $f \supset g$. This poset has the countable chain condition, and so all cardinals are preserved. If G is a generic filter, set $f^* = \bigcup_{f \in G} f$. Since G is a filter, f^* is a function, and by a density argument the domain of f^* is ω. Then $a := \{n < \omega \mid f^*(n) = 1\}$ is a new real.
2. The Cohen poset to add λ many reals, $Add(\omega, \lambda) := \{f : \lambda \times \omega \to \{0, 1\} \mid |\operatorname{dom}(f)| < \omega\}$, ordered by reverse inclusion. It also has the countable chain condition, and so all cardinals are preserved. If G is a generic filter, $f^* := \bigcup_{f \in G} f$ is a total function. Let $a_\alpha := \{n < \omega \mid f^*(\alpha, n) = 1\}$ for $\alpha < \lambda$. Then $\langle a_\alpha \mid \alpha < \lambda \rangle$ are distinct new reals.
3. The Cohen poset to add λ many new subsets of a regular cardinal κ, $Add(\kappa, \lambda) := \{f : \lambda \times \kappa \to \{0, 1\} \mid |\operatorname{dom}(f)| < \kappa\}$, ordered by reverse inclusion. It has the κ^+-chain condition and it is κ-closed. So, all cardinals are preserved. As before, if G is a generic filter,

$f^* := \bigcup_{f \in G} f$ is a total function and let $a_\alpha := \{\eta < \kappa \mid f^*(\alpha, \eta) = 1\}$ for each $\alpha < \lambda$. Then $\langle a_\alpha \mid \alpha < \lambda \rangle$ are distinct new subsets of κ.

(4) Suppose $\kappa < \lambda$ are cardinals, such that κ is regular and $\lambda^{<\kappa} = \lambda$. The Levy collapse $Col(\kappa, \lambda) := \{f : \kappa \to \lambda \mid |\mathrm{dom}(f)| < \kappa\}$, ordered by reverse inclusion, is the poset to collapse λ to κ. It is κ-closed and has the λ^+-chain condition. So, cardinals $\leq \kappa$ and $\geq \lambda^+$ are preserved. If G is a generic filter, then $f^* := \bigcup_{f \in G} f : \kappa \to \lambda$ is a total onto function. So, in $V[G]$, $|\lambda| = \kappa$.

(5) Suppose κ is a regular cardinal and λ is inaccessible. The Levy collapse $Col(\kappa, < \lambda) := \{f : \lambda \times \kappa \to \lambda \mid |\mathrm{dom}(f)| < \kappa, (\forall \alpha < \lambda, \eta < \kappa)(f(\alpha, \eta) < \alpha)\}$, ordered by reverse inclusion, is the poset to collapse every cardinal below λ to κ. It is κ-closed and has the λ-chain condition. So, cardinals $\leq \kappa$ and $\geq \lambda$ are preserved. If G is a generic filter, let $f^* = \bigcup_{f \in G} f$. Then for each $\alpha < \lambda$, let $f^*_\alpha : \kappa \to \alpha$ be given by $f^*_\alpha(\eta) = f^*(\alpha, \eta)$; it is a total onto function. So, in $V[G]$, $\lambda = \kappa^+$, i.e. it is the cardinal successor of κ.