DESCRIPTIVE SET THEORY PROBLEM SET

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1. Let X be a second-countable topological space.

(a) Show that X has at most continuum many open subsets.

(b) Let α, β, γ denote ordinals. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

Hint: Use the same idea as in the proof of (a).

(c) Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ open subsets of X eventually stabilizes, i.e. there is $\gamma < \omega_1$ such that for all $\alpha < \omega_1$ with $\alpha \geq \gamma$, we have $U_\alpha = U_\gamma$.

Hint: Use the regularity of ω_1.

(d) Conclude that parts (a), (b) and (c) are also true for closed sets.

2. Prove that any separable metric space has cardinality at most continuum.

Remark: This is true more generally for first-countable separable Hausdorff topological spaces, but false for general separable Hausdorff topological spaces (try to construct a counter-example).

3. (a) Show that a metric space X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n\in\mathbb{N}}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton).

(b) Show that the requirement in (a) that $\operatorname{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed **balls** with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.

Hint: Use $\mathbb{N}$ as the underlying set for your metric space.

4. By definition, the class of G_δ sets is closed under countable intersections. Show that it is also closed under finite unions. Equivalently, the class of F_σ sets is closed under finite intersections.

Hint: Think in terms of quantifiers $\forall$ and $\exists$ rather than intersections and unions; for example, if $A = \bigcap_n U_n$, then $x \in A \iff \forall n(x \in A_n)$.

Many of the problems here are borrowed from "Classical Descriptive Set Theory" by A. Kechris.

5. (a) Show that the Cantor set (with relative topology of $\mathbb{R}$) is homeomorphic to the Cantor space.

(b) Show that the Baire space $\mathcal{N}$ is homeomorphic to a G_δ subset of the Cantor space $\mathcal{C}$.

(c) Show that the set of irrationals (with the relative topology of $\mathbb{R}$) is homeomorphic to the Baire space.

HINT: Use the continued fraction expansion.

6. Let $T \subseteq A^{<\mathbb{N}}$ be a tree and suppose it is finitely branching. Prove that $[T]$ is compact.

7. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T doesn't have an infinite branch.

8. Let S, T be trees on sets A, B, respectively. Prove that for any continuous function $f : [S] \to [T]$ there is a monotone map $\varphi : S \to T$ such that $f = \varphi^*$.

HINT: For $s \in S$, define $\varphi(s)$ to be the longest $t \in T$ such that $|t| \leq |s|$ and $N_t \supseteq f(N_s)$.

9. Using the outline below, prove the following:

Proposition. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is compact.*

(2) *Every sequence in X has a convergent subsequence.*

(3) *X is complete and totally bounded.*

In particular, compact metrizable spaces are Polish.

(1) $\Rightarrow$ (2): For a sequence $(x_n)_n$, let K_m be the closure of the tail $\{x_n\}_{n \geq m}$ of the sequence and use the intersection-of-closed sets version of the definition of compactness.

(2) $\Rightarrow$ (3): For total boundedness, fix an $\varepsilon > 0$ and start constructing an ε-net F by adding elements to your F that are not yet covered by $B(F, \varepsilon)$. For completeness, note that if a subsequence of a Cauchy sequence converges, then so does the entire sequence.

(3) $\Rightarrow$ (2): Let $(x_n)_n$ be a sequence in X. Think of the x_n-s as pigeons and a finite ε-net as holes.

(2) and (3) $\Rightarrow$ (1): Somewhat trickier, look up Theorem 0.25 in Folland's "Real Analysis".

10. Let X be a compact metric space and Y be a separable complete metric space. Let $C(X, Y)$ be the space of continuous functions from X to Y equipped with the uniform metric, i.e. for $f, g \in C(X, Y)$,

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Prove that $C(X, Y)$ is a separable complete metric space, hence Polish.

Hint 1: Proving separability is tricky, so you may want to first prove it for $X = [0,1]$ and $Y = \mathbb{R}$. In the general case (to prove separability), note that by uniform continuity,

$$C(X,Y) = \bigcup_n A_{n,m},$$

for every $n \in \mathbb{N}$, where

$$A_{n,m} = \{f \in C(X,Y) : \forall x, y \in X (d_X(x,y) < 1/n \Rightarrow d_Y(f(x), f(y)) < 1/m)\}.$$

Realize that it is enough to show that for any $n, m \in \mathbb{N}$, there is a countable $B_{n,m} \subseteq A_{n,m}$ such that for any $f \in A_{n,m}$ there is $g \in B_{n,m}$ with $d_u(f,g) < 3/m$. Now fix n, m and try to construct $B_{n,m}$; when doing so, don't try to *define* each function in $B_{n,m}$ by hand as you would maybe do in the case $X = [0,1]$; instead, carefully *pick* them out of functions in $A_{n,m}$.

Hint 2: This is Theorem 4.19 in Kechris's "Classical Descriptive Set Theory".

11. Show that Hausdorff metric on $\mathcal{K}(X)$ is compatible with the Vietoris topology.

12. Let (X,d) be a metric with $d \leq 1$. For $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$ and nonempty $K \in \mathcal{K}(X)$:

(a) $\delta(K, K_n) \to 0 \Rightarrow K \subseteq \underline{\mathrm{T}\lim}_n K_n$;

(b) $\delta(K_n, K) \to 0 \Rightarrow K \supseteq \overline{\mathrm{T}\lim}_n K_n$.

In particular, $d_H(K_n, K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. Show that the converse may fail.

13. Let (X,d) be a compact metric with $d \leq 1$. For sequence $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$, show the following:

(a) if $\underline{\mathrm{T}\lim}_n K_n \neq \emptyset$ then $\delta(\underline{\mathrm{T}\lim}_n K_n, K_m) \to 0$ as $m \to \infty$;

(b) $\delta(K_m, \overline{\mathrm{T}\lim}_n K_n) \to 0$ as $m \to \infty$.

So if $K = \mathrm{T}\lim_n K_n$ exists, $d_H(K_n, K) \to 0$.

14. Let (X,d) be a metric space with $d \leq 1$. Then $x \mapsto \{x\}$ is an isometric embedding of X into $\mathcal{K}(X)$.

15. Let (X,d) be a metric space with $d \leq 1$ and assume $K_n \to K$. Then any sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in K_n$ has a subsequence converging to a point in K.

16. Let X be metrizable.

(a) The relation "$x \in K$" is closed, i.e. $\{(x, K) : x \in K\}$ is closed in $X \times \mathcal{K}(X)$.

(b) The relation "$K \subseteq L$" is closed, i.e. $\{(K, L) : K \subseteq L\}$ is closed in $\mathcal{K}(X)^2$.

(c) The relation "$K \cap L \neq \emptyset$" is closed, i.e. $\{(K, L) : K \cap L \neq \emptyset\}$ is closed in $\mathcal{K}(X)^2$.

(d) The map $(K, L) \mapsto K \cup L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is continuous.

(e) If Y is metrizable, then the map $(K, L) \mapsto K \times L$ from $\mathcal{K}(X) \times \mathcal{K}(Y)$ into $\mathcal{K}(X \times Y)$ is continuous.

(f) Find a compact X for which the map $(K, L) \mapsto K \cap L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is not continuous.

17. Let X be a topological space.

(a) If X is nonempty perfect, then so is $\mathcal{K}(X) \setminus \{\emptyset\}$.

(b) If X is compact metrizable, then $C(X)$ is perfect, where $C(X) = C(X, \mathbb{R})$.

18. (AC) Show that any nonempty perfect compact Hausdorff space X has cardinality at least continuum by constructing an injection from the Cantor space into X.

Hint: Mimic the proof for Polish spaces.

19. Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. Conclude that $\mathbb{Q}$ is not Polish in the relative topology of $\mathbb{R}$.

20. Show that the perfect kernel of a Polish space X is the largest perfect subset of X, i.e. it contains all other perfect subsets.

21. A *topological group* is a group with a topology on it so that group multiplication $(x, y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Show that a countable topological group is Polish if and only if it is discrete.

22. Let X be separable metrizable. Show that

$$\mathcal{K}_p(X) = \{K \in \mathcal{K}(X) : K \text{ is perfect}\}$$

is a G_δ set in $\mathcal{K}(X)$. In particular, if X is Polish, then so is $\mathcal{K}_p(X)$.

23. (a) Let X be a Polish space. Show that if $K \subseteq X$ is countable and compact, then its Cantor-Bendixson rank $|K|_C$ is not a limit ordinal.

(b) For each non-limit ordinal $\alpha < \omega_1$, construct a countable compact subset K_α of $\mathcal{C}$, whose Cantor-Bendixson rank is exactly α.

24. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree.

(a) Suppose T is pruned. Find a condition (on the nodes of the tree T) such that T satisfies it if and only if $[T]$ is perfect (as a subset of $\mathcal{N}$).

(b) Define a Cantor-Bendixson derivative T' of T, as well as the iterated derivatives $(T^\alpha)_{\alpha \in \mathrm{ON}}$, such that $[T^\infty]$ is the perfect kernel of $[T]$, i.e. $[T^\infty] = [T]^\infty$.

Remark: The statement of this problem is somewhat vague and informal, but understanding it is part of the challenge.

25. Let X be a second countable zero-dimensional space.

(a) Prove Kuratowski's reduction property: If $A, B \subseteq X$ are open, there are open $A^* \subseteq A, B^* \subseteq B$ with $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \emptyset$.

Hint: Write A and B as countable unions of clopen sets: $A = \bigcup_n A_n$, $B = \bigcup_n B_n$. Put those points x of A in A^* that are covered by A_n no later than by B_n, i.e. if n is the smallest number such that $x \in A_n \cup B_n$, then $x \in A_n$.

(b) Conclude the following separation property: For any disjoint closed sets $A, B \subseteq X$, there is a clopen set C separating A and B, i.e. $A \subseteq C$ and $B \cap C = \emptyset$.

26. (a) Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and prove that there is a Luzin scheme $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with vanishing diameter and satisfying the following properties:

(i) $A_\emptyset = X$;

(ii) A_s is nonempty clopen;

(iii) $A_s = \bigcup_{i \in \mathbb{N}} A_{s^\frown i}$.

HINT: Assuming A_s is defined, cover it by countably many clopen sets of diameter at most $\delta < 1/n$, and choose the δ small enough so that any such cover is necessarily infinite.

(b) Derive the Alexandrov–Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis of (a).

27. For this exercise, you may use the Alexandrov–Urysohn theorem without proof.

(a) Let $Y \subseteq \mathbb{R}$ be G_δ and such that Y, $\mathbb{R} \setminus Y$ are dense in $\mathbb{R}$. Show that Y is homeomorphic to $\mathcal{N}$.

(b) Show that part (a) may fail if $\mathbb{R}$ is replaced by $\mathbb{R}^2$.

(c) However, prove that part (a) holds if $\mathbb{R}$ is replaced by any zero-dimensional nonempty Polish space.

28. Show that for any Polish space X there is a continuous open surjection $g : \mathcal{N} \twoheadrightarrow X$ by constructing a sequence $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of open subsets of X such that

(i) $U_\emptyset = X$

(ii) $\overline{U}_{s^\frown i} \subseteq U_s$

(iii) $U_s = \bigcup_i U_{s^\frown i}$

(iv) $\operatorname{diam}(U_s) < 2^{-|s|}$.

CAUTION: We don't require $U_{s^\frown i} \cap U_{s^\frown j} = \emptyset$ for $i \neq j$ (which makes your life easy), so the associated map g may not be injective.

29. Using part (c) of Problem 27, prove the following:[1]

Theorem (Strengthening of the Perfect Set Theorem). *Every nonempty perfect Polish space contains a dense G_δ subset homeomorphic to the Baire space.*

30. The following steps outline a proof of the Baire category theorem for locally compact Hausdorff spaces.

1) Show that compact Hausdorff spaces are normal.

[1] Thanks to Anton Bernshteyn for suggesting this problem.

2) Using part (1), prove that in locally compact[2] Hausdorff space X, for every nonempty open set U and every point $x \in U$, there is a nonempty precompact[3] open $V \ni x$ with $\overline{V} \subseteq U$.

3) Prove that locally compact Hausdorff spaces are Baire.

31. For topological space X, Y, a continuous map $f : X \to Y$ is called *category-preserving* if f-preimages of meager sets are meager.

(a) Show that any continuous open map $f : X \to Y$ is category-preserving (in fact, f-preimages of nowhere dense are nowhere dense). In particular, projections are category-preserving.

(b) For topological spaces X, Y, if X is Baire, then, for a continuous map $f : X \to Y$, the following are equivalent:

(1) f is category preserving.

(2) f-preimages of nowhere dense sets are nowhere dense.

(3) f-preimages of dense open sets are dense.

32. [4]

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational but discontinuous at every rational.

(b) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational but discontinuous at every irrational.

33. Recall that $C([0, 1])$ is a Polish space with the uniform metric. Show that the generic element of $C([0, 1])$ is nowhere differentiable following the outline below.

1) Prove that given $m \in \mathbb{N}$, any function $f \in C([0, 1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0, 1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geq m$.

2) For each $n \geq 1$, let E_n be the set of all functions $f \in C([0, 1])$, for which there is $x_0 \in [0, 1]$ (depending on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$. Show that E_n is nowhere dense using the fact that if g is as in (1) with $m = 2n$, then some open neighborhood of g is disjoint from E_n.

34. Let X be a perfect Polish space and show that a generic compact subset of X is perfect, i.e. show that the set $\mathcal{K}_p(X)$ is comeager in $\mathcal{K}(X)$ (see Problem 22).

35. A *finite bounded game* on a set A is a game similar to infinite games, but the players play at most n number of steps before the winner is decided, for some fixed number

[2] A topological space is said to be *locally compact* if every point has a neighborhood basis that consists of precompact[3] open sets.

[3] *Precompact* sets are those contained in compact sets. For Hausdorff spaces, this is equivalent to having a compact closure.

[4] Thanks to Francesco Cellarosi for bringing up the statements of this problem to me.

$n \geq 1$ (say a million). More formally, the game is a tree $T \subseteq A^{<n}$, for some n, and the runs of the game are exactly the elements of the set $\text{Leaves}(T)$ of all leaves of T, so the payoff set is a subset $D \subseteq \text{Leaves}(T)$. Player I wins the run $s \in \text{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$. All games that appear in real life are such games, e.g. chess (counting ties as a win for Player II).

Prove the determinacy of finite bounded games.

Hint: Let's write down what it means for Player I to have a winning strategy in this game, assuming for simplicity that n is even and that all of the runs of the game are of length exactly n:

$$\exists a_1 \forall a_2 ... \exists a_{n-1} \forall a_n ((a_1, ..., a_n) \in D).$$

What happens when you negate this statement?

36. A *finite game* on a set A is a game similar to infinite games, but the players play only finitely many steps before the winner is decided. More formally, it is a (possibly infinite) tree $T \subseteq A^{<\mathbb{N}}$ that has no infinite branches, and the set of runs is $\text{Leaves}(T)$, so the payoff set is a subset $D \subseteq \text{Leaves}(T)$. Player I wins the run $s \in \text{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$.

(a) Prove the determinacy of finite games.

Hint: Call a position $s \in T$ determined, if from that point on, one of the players has a winning strategy. Thus, no player has a winning strategy in the beginning iff $\emptyset$ is undetermined. What can you say about extensions of undetermined positions?

(b) Conclude the determinacy of clopen infinite games (i.e. the games defined in class, where the runs are elements of $A^{\mathbb{N}}$).

37. In ZF (in particular, don't use AC or $\neg$AD), define a game with rules $G(T, D)$ on the set $A = \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ (i.e. define a pruned tree $T \subseteq A^{<\mathbb{N}}$ and a set $D \subseteq A^{\mathbb{N}}$), so that ZF+$\neg$AD implies that this game is undetermined. In other words, you have to define the tree T and the payoff set D without using $\neg$AD, but then prove that the game $G(T, D)$ is undetermined using $\neg$AD.

Hint: Note that besides playing subsets of $\mathbb{N}^{\mathbb{N}}$, players can also play natural numbers in the sense that $\mathbb{N} \hookrightarrow \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ by $n \mapsto \{(n)_{i \in \mathbb{N}}\}$.

38. Let X be a second countable Baire space. Show that the σ-ideal $\text{MGR}(X)$ has the countable chain condition in $\text{BP}(X)$, i.e. there is no uncountable family $\mathcal{A} \subseteq \text{BP}(X)$ of non-meager sets such that for any two distinct $A, B \in \mathcal{A}$, $A \cap B$ is meager.

39. Let X be a topological space.

(a) If $A_n \subseteq X$, then for any open $U \subseteq X$,

$$U \Vdash \bigcap_n A_n \iff \forall n (U \Vdash A_n).$$

(b) If X is a Baire space, A has the BP in X and $U \subseteq X$ is nonempty open, then

$$U \Vdash A^c \iff \forall V \subseteq U (V \nVdash A),$$

where V varies over a weak basis[5] for X.

(c) If X is a Baire space, the sets $A_n \subseteq X$ have the BP, and U is nonempty open, then

$$U \Vdash \bigcup_n A_n \iff \forall V \subseteq U \exists W \subseteq V \exists n (W \Vdash A_n).$$

where V, W vary over a weak basis for X.

40. Prove that a topological group G is Baire iff G is non-meager.

41. Let X be a topological space and $A \subseteq X$.

(a) Show that $U(A)$ is regular open, i.e. it is equal to the interior of its closure.

(b) If moreover X is a Baire space and A has the BP, then $U(A)$ is the unique regular open set U with $A =^* U$.

42. Let G be a Polish group (i.e. a topological group whose topology happens to be Polish) and let $H < G$ be a subgroup. Prove that H is Polish iff H is closed.

Hint: Consider H inside $\overline{H}$. What is the Baire category status (meager/non-meager/comeager) of H inside $\overline{H}$? If $H \subsetneq \overline{H}$, look at the cosets.

43. Let Γ be a group acting on a Polish space X by homeomorphisms (i.e. each element $\gamma \in \Gamma$ acts as a homeomorphism of X). A set $A \subseteq X$ is called invariant if $\gamma A = A$ for all $\gamma \in \Gamma$. The action $\Gamma \curvearrowright X$ is called *generically ergodic* if every invariant set $A \subseteq X$ with the BP is either meager or comeager. For a set $A \subseteq X$, denote by $[A]_\Gamma$ the saturation of A, namely $[A]_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma A$.

Prove that the following are equivalent:

(1) $\Gamma \curvearrowright X$ is generically ergodic.

(2) Every invariant nonempty open set is dense.

(3) For comeager-many $x \in X$, the orbit $[x]_\Gamma$ is dense.

(4) There is a dense orbit.

(5) For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.

Hint: For (2)$\Rightarrow$(3), take a countable basis $\{U_n\}_{n \in \mathbb{N}}$ and consider $\bigcap_n [U_n]_\Gamma$.

44. Show that the Kuratowski-Ulam theorem fails if A does not have the BP by constructing a non-meager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line.

Hint: Note that there are only continuum many F_σ sets, so take a transfinite enumeration $(F_\xi)_{\xi < 2^{\aleph_0}}$ of all *meager* F_σ sets, and construct a sequence $(a_\xi)_{\xi < 2^{\aleph_0}}$ of points in $\mathbb{R}^2$ by transfinite recursion so that for each $\xi < 2^{\aleph_0}$,

$$\{a_\lambda : \lambda \leq \xi\} \nsubseteq F_\xi,$$

and no three of the points in $\{a_\lambda : \lambda \leq \xi\}$ lie on a straight line.

[5] A *weak basis* for a topological space X is a collection $\mathcal{V}$ of nonempty open sets such that every nonempty open set $U \subseteq X$ contains at least one $V \in \mathcal{V}$.

Hint: Recall that in perfect Polish spaces (such as $\mathbb{R}, \mathbb{R}^2$), any non-meager subset with the BP contains a copy of the Cantor space (this is because it contains a non-meager G_δ set). Now if $A := \{a_\lambda : \lambda \le \xi\} \subseteq F_\xi$, apply Kuratowski-Ulam to F_ξ to find $x \in \mathbb{R}$ such that $(F_\xi)_x$ is meager and the vertical line $L_x = \{(x, y) \in \mathbb{R} : y \in \mathbb{R}\}$ is disjoint from A.

45. Show that if X, Y are second countable Baire spaces, so is $X \times Y$.

46. **Definition.** A *filter* on a set X is a set $\mathcal{U} \subseteq \mathscr{P}(X)$ such that

(i) (Nontriviality) $X \in \mathcal{U}$ but $\emptyset \notin \mathcal{U}$;

(ii) (Upward closure) $A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$;

(iii) (Closure under finite intersections) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$.

A filter $\mathcal{U}$ is called an *ultrafilter* if $A \notin \mathcal{U} \Rightarrow A^c \in \mathcal{U}$ for every $A \subseteq X$. Finally, an ultrafilter is called *principal* if for some $x \in X$, $\{x\} \in \mathcal{U}$ (or, equivalently, $\mathcal{U} = \{A \subseteq X : x \in A\}$).

It is useful to think of a filter $\mathcal{U}$ as the family of all conull sets of a $\{0,1\}$-valued finitely additive measure $\mu_{\mathcal{U}}$ on a subalgebra of $\mathscr{P}(X)$. In other words, sets in $\mathcal{U}$ should be thought of as large sets. $\mathcal{U}$ being an ultrafilter simply means that $\mu_{\mathcal{U}}$ is defined on all of $\mathscr{P}(X)$; in other words, if a set is not large then it is small (i.e. there are no intermediate sets). Also, $\mathcal{U}$ being principal means that $\mu_{\mathcal{U}}$ is a Dirac measure (i.e. a point-mass at some point x).

(a) (AC) Prove that for every infinite set X, there exists a non-principal ultrafilter on X; do it by showing that every filter is contained in an ultrafilter and applying this to the filter of cofinite sets (called the Fréchet filter).

(b) Identifying $\mathscr{P}(\mathbb{N})$ with $\mathcal{C} = 2^{\mathbb{N}}$, view ultrafilters on $\mathbb{N}$ as subsets of $\mathcal{C}$ and show that no non-principal ultrafilter $\mathcal{U}$ has the BP (as a subset of $\mathcal{C}$).

47. Using the outline below, prove Pettis's theorem:

Theorem (Pettis)**.** *Let G be a topological group and $A \subseteq G$ have the BP. If A is non-meager, then $A^{-1}A$ contains an open neighborhood of the identity 1_G; in fact if $U \Vdash A$, then $U^{-1}U \subseteq A^{-1}A$.*

1) By Problem 40, G must be Baire.

2) Note that for any sets $B, C \subseteq G$,

$$B \subseteq C^{-1}C \iff \forall h \in B\ (Ch \cap C \neq \emptyset). \qquad (*)$$

3) Let $U \subseteq G$ be nonempty open such that $U \Vdash A$. Fix arbitrary $g \in U$ and note that $V := g^{-1}U \subseteq U^{-1}U$ is an open neighborhood of 1_G. Thus, by $(*)$, $\forall h \in V$, $Uh \cap U \neq \emptyset$.

4) Conclude that for each $h \in V$, $Ah \cap A \neq \emptyset$, and hence, by $(*)$ again, $V \subseteq A^{-1}A$.

5) Note that we have shown $g^{-1}U \subseteq A^{-1}A$ for arbitrary $g \in U$, and thus, $U^{-1}U \subseteq A^{-1}A$.

48. Let G be a Baire topological group (i.e. G is non-meager) and let $H < G$ be a subgroup with the BP. Prove if H is non-meager then it is actually clopen! In particular, if H has countable index in G, then it is clopen.

49. (a) **Automatic continuity:** Let G, H be topological groups, where G is Baire and H is separable. Then every Baire measurable group homomorphism $\varphi : G \to H$ is actually continuous!

HINT: Enough to prove continuity at 1_G, so let $U \ni 1_H$ be open and take an open neighborhood $V \ni 1_H$ such that $V^{-1}V \subseteq U$. Using the separability of H, deduce that $\varphi^{-1}(hV)$ is non-meager for some $h \in H$ and apply Pettis's theorem.

(b) Conclude that if $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a Baire measurable group homomorphism, then for some $a \in \mathbb{R}$, $f(x) = ax$ for all $x \in \mathbb{R}$.

HINT: First show this for integers, then for rationals, etc.

50. Prove the following facts about the density topology on $\mathbb{R}$. (Below λ denotes the Lebesgue measure on $\mathbb{R}$ and all topological terms are with respect to the density topology.)

(a) Every nonempty open set has positive measure.

(b) For a Lebesgue measurable set $A \subseteq \mathbb{R}$, explicitly compute $\mathrm{Int}(A)$ and $\overline{A}$, and conclude that $\lambda(\mathrm{Int}(A)) = \lambda(A) = \lambda(\overline{A})$.

51. Consider $\mathbb{R}$ with the density topology and Lebesgue measure λ. For $A \subseteq \mathbb{R}$, prove that the following are equivalent:

(1) A is nowhere dense in the density topology;

(2) A is meager in the density topology;

(3) A is λ-null.

Conclude that A has the BP in the density topology if and only if it is Lebesgue measurable.

52. Let $X = \mathbb{I}^{\mathbb{N}}$ and put $C_0 = \{(x_n)_{n \in \mathbb{N}} : x_n \to 0\}$. Show that C_0 is in $\mathbf{\Pi}^0_3(X)$.

53. Let X be a topological space, $Y \subseteq X$, and let ξ be an ordinal with $1 \leq \xi < \omega_1$. Prove the following:

(a) If $\mathbf{\Gamma}$ is one of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathcal{B}$, then $\mathbf{\Gamma}(Y) = \mathbf{\Gamma}(X)|_Y := \{A \cap Y : A \in \mathbf{\Gamma}(X)\}$.

(b) We also always have $\mathbf{\Delta}^0_\xi(Y) \supseteq \mathbf{\Delta}^0_\xi(X)|_Y$. If moreover, $Y \in \mathbf{\Delta}^0_\xi(X)$, then we also have $\mathbf{\Delta}^0_\xi(Y) \subseteq \mathbf{\Delta}^0_\xi(X)|_Y$. However, give an example of a Polish space X and $Y \subseteq X$ such that the last inclusion is false for $\xi = 1$.

54. A class $\mathbf{\Gamma}$ of sets is called self-dual if it is closed under complements, i.e. $\check{\mathbf{\Gamma}} = \mathbf{\Gamma}$. Show that if $\mathbf{\Gamma}$ is a self-dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set for

$\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$, can have X-universal sets.

55. Letting X be a separable metrizable space and $\lambda < \omega_1$ be a limit ordinal, put

$$\mathbf{\Omega}^0_\lambda(X) := \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \; (= \bigcup_{\xi<\lambda} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(X)).$$

(a) Let Y be an uncountable Polish space and prove that there exists a set $P \in \mathbf{\Delta}^0_\lambda(Y \times X)$ that parameterizes $\mathbf{\Omega}^0_\lambda(X)$.

Hint: First construct such a set for $Y = \mathbb{N} \times \mathcal{C}$. Then conclude it for $Y = \mathcal{C}$ using the fact that the following functions are continuous: $()_0 : \mathcal{C} \to \mathbb{N}$ and $()_1 : \mathcal{C} \to \mathcal{C}$ defined for $y \in \mathcal{C}$ by

$$y = 1^{(y)_0} {}^\frown 0 {}^\frown (y)_1.$$

Finally, conclude the statement for any Y using the perfect set property.

(b) Conclude that if X is uncountable Polish, then $\mathbf{\Delta}^0_\lambda(X) \supsetneq \mathbf{\Omega}^0_\lambda(X)$.

56. Let X, Y be topological spaces and let $\mathrm{proj}_X : X \times Y \to X$ be the projection function. Prove the following statements:

(a) proj_X is continuous and open.

(b) proj_X does not in general map closed sets to closed sets, even for $X = Y = \mathbb{R}$.

Remark: We will see shortly in the course that for certain $Y = \mathcal{N}$, the projection of a closed set may not even be Borel in general.

(c) For $X = Y = \mathbb{R}$ (or in general any σ-compact Hausdorff space), proj_X maps closed sets to σ-compact (and hence F_σ) sets.

(d) **Tube lemma:** If Y is compact, then proj_X indeed maps closed sets to closed sets.

Hint: It is perhaps tempting to use sequences, but this would only work for first-countable spaces. Instead, use the open cover definition of compact and show that for closed $F \subseteq X \times Y$, every point $x \in X \setminus \mathrm{proj}_X(F)$ has an open neighborhood disjoint from $\mathrm{proj}_X(F)$. The "correct" solution should use nothing but definitions.

57. (a) Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets, i.e. for a given Polish space $(X, \mathcal{T})$, there exists a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.

(b) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be Polish and $f : X \to Y$ a Borel isomorphism. Show that there are Polish topologies $\mathcal{T}'_X \supseteq \mathcal{T}_X, \mathcal{T}'_Y \supseteq \mathcal{T}_Y$ with $\mathcal{B}(\mathcal{T}'_X) = \mathcal{B}(\mathcal{T}_X)$, $\mathcal{B}(\mathcal{T}'_Y) = \mathcal{B}(\mathcal{T}_Y)$ such that $f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}'_Y)$ is a homeomorphism. Moreover, $\mathcal{T}'_X, \mathcal{T}'_Y$ can be taken to be zero-dimensional.

(c) Let G be a countable group and consider a Borel action of G on a Polish space $(X, \mathcal{T})$, i.e. each $g \in G$ acts as a Borel automorphism of X. Prove that there exists a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$ that makes the action of G continuous. Moreover, $\mathcal{T}_0$ can be taken to be zero-dimensional.

58. Show that the class of analytic sets is closed under

(a) continuous preimages,
(b) continuous images,
(c) countable unions,
(d) countable intersections.

Can we replace "continuous" by "Borel" above?

59. Let X be Polish and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint analytic sets in X. Prove that there are disjoint Borel sets $\{B_n\}_{n\in\mathbb{N}}$ with $B_n \supseteq A_n$.

60. Let X, Y be Polish and $f : X \to Y$ Borel. Show that for $A \subseteq f(X)$, if $f^{-1}(A)$ is Borel, then A is Borel relative to $f(X)$, i.e. there is a Borel $A' \subseteq Y$ such that $A = A' \cap f(X)$.

61. Let X be Polish and let E be an analytic equivalence relation on X (i.e. $E \subseteq X^2$ is analytic).

(a) Show that for an analytic set A, its saturation $[A]_E = \{x \in X : \exists y \in A(xEy)\}$ is also analytic.
(b) Let $A, B \subseteq X$ be disjoint invariant analytic sets (i.e. $[A]_E = A$, $[B]_E = B$). Prove that there is an invariant Borel set D separating A and B, i.e. $D \supseteq A$ and $D \cap B = \emptyset$.

62. Construct an example of a closed equivalence relation E on a Polish space X and a closed set $C \subseteq X$ such that the saturation $[C]_E$ is analytic but not Borel.

Remark: This shows that in part (a) of the previous problem, "analytic" is the best we can hope for.

Hint: Take analytic $A \subseteq \mathcal{N}$ that's not Borel and let $C \subseteq \mathcal{N}^2$ be a closed set projecting down onto A. Define an appropriate equivalence relation E on $\mathcal{N}^2$ (i.e. $E \subseteq \mathcal{N}^2 \times \mathcal{N}^2$).

63. Let X be set and let $\mathcal{T}, \mathcal{T}'$ be Polish topologies on X such that $\mathcal{T} \subseteq \mathcal{B}(\mathcal{T}')$ (for example, this would hold if $\mathcal{T} \subseteq \mathcal{T}'$). Show that $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}')$.

64. Prove the following characterization of Borel sets: A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$.

65. Prove that any standard Borel space $(X, \mathcal{S})$ admits a Borel linear ordering, i.e. there is a linear ordering $<$ of X such that $<$ is Borel as a subset of X^2 (with respect to the product σ-algebra).

66. Let X be Polish and consider the coding map $c : \mathcal{F}(X) \to 2^{\mathbb{N}}$ defined by $F \mapsto$ the characteristic function of $\{n \in \mathbb{N} : F \cap U_n \neq \emptyset\}$. Prove that for $x \in 2^{\mathbb{N}}$, $x \in c(\mathcal{F}(X))$ if and only if

$$\forall U_n \subseteq U_m [x(n) = 1 \to x(m) = 1]$$

and

$$\forall U_n \forall \varepsilon \in \mathbb{Q}^+ \big[x(n) = 1 \to \exists \overline{U_m} \subseteq U_n \text{ with } \operatorname{diam}(U_m) < \varepsilon \text{ such that } x(m) = 1)\big].$$

Conclude that $c(\mathcal{F}(X))$ is a G_δ subset of $2^{\mathbb{N}}$ and hence the Effros space $\mathcal{F}(X)$ is standard Borel.

67. Let X be a Polish space. Show that $\mathcal{K}(X)$ is a Borel subset of $\mathcal{F}(X)$.[6]

68. Let X be a Polish space. A function $s : \mathcal{F}(X) \to X$ is called a selector if $s(F) \in F$ for every nonempty $F \in \mathcal{F}(X)$. The goal of this problem is to show that for every Polish space X, the Effros Borel space $\mathcal{F}(X)$ admits a Borel selector.

(a) Show that $\mathcal{F}(\mathcal{N})$ admits a Borel selector.

(b) By Problem 28, there is a continuous open surjection $g : \mathcal{N} \to X$. Prove that the map $f : \mathcal{F}(X) \to \mathcal{F}(\mathcal{N})$ defined by $F \mapsto g^{-1}(F)$ is Borel.

(c) Conclude that $\mathcal{F}(X)$ admits a Borel selector.

69. Let X, Y be Polish spaces and let $f : X \to Y$ be a continuous function such that $f(X)$ is uncountable. Put

$$\mathcal{K}_f(X) = \{K \in \mathcal{K}(X) : f|_K \text{ is injective}\},$$

and note that, for a fixed countable basis $\mathcal{U}$ of X and for $K \in \mathcal{K}(X)$,

$$K \in \mathcal{K}_f(X) \iff \forall U_1, U_2 \in \mathcal{U} \text{ with } \overline{U_1} \cap \overline{U_2} = \emptyset [f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset].$$

Next, show that for fixed $U_1, U_2 \in \mathcal{U}$ with $\overline{U_1} \cap \overline{U_2} = \emptyset$ the set

$$\mathcal{V} = \big\{K \in \mathcal{K}(X) : f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset\big\}$$

is open in $\mathcal{K}(X)$, and hence $\mathcal{K}_f(X)$ is G_δ.

70. Let X be a Polish space, $F \subseteq X \times \mathcal{N}$ and $A = \operatorname{proj}_X(F)$. Show that if Player II has a winning strategy in the unfolded Banach-Mazur game $G^{**}(F, X)$, then A is meager.

71. For a topological space X, show that $\mathrm{BP}(X)$ admits envelopes: for a given $A \subseteq X$, first find a $\mathrm{BP}(X)$-envelope for it in terms of $U(\cdot)$, then write down explicitly what the set is.

72. Let X be a Polish space and let $\mathbf{C}(X)$ denote the smallest σ-algebra on X containing $\mathcal{B}(X)$ and closed under the operation $\mathcal{A}$.

(a) Show that $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X) \subseteq \mathbf{C}(X)$.

HINT: For $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X)$, it is enough to show that $\mathcal{A}\mathbf{\Pi}^1_1(X)$ is closed under countable unions and countable intersections. For countable unions, use the natural bijection $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ given by $(n, s) \mapsto n^\frown s$. For countable intersections, use the usual diagonal (snake-like) bijection $\mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$ to monotonically encode finite sequences of elements of $\mathbb{N}^{<\mathbb{N}}$ into single elements of $\mathbb{N}^{<\mathbb{N}}$.

[6]Thanks to Anton Bernshteyn for suggesting this problem.

(b) For each uncountable Polish space Y show that there is a Y-universal set for $\mathcal{A}\mathbf{\Pi}^1_1(X)$.

HINT: Enough to prove for $Y = \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}}$ (why?). Start with a $\mathcal{N}$-universal set $F \subseteq \mathcal{N} \times X$ for $\mathbf{\Pi}^1_1(X)$ and for each $s \in \mathbb{N}^{<\mathbb{N}}$, consider the set $P_s \subseteq \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$ defined as follows: for $(y, x) \in \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$, put $(y, x) \in P_s :\Leftrightarrow (y(s), x) \in F$.

(c) Conclude that for uncountable X, $\sigma(\mathbf{\Sigma}^1_1(X)) \subsetneq \mathcal{A}\mathbf{\Pi}^1_1(X) \subsetneq \mathbf{C}(X)$.

73. (Fun problem) Prove directly (without using Wadge's theorem or lemma) that any countable dense $Q \subseteq 2^{\mathbb{N}}$ is $\mathbf{\Sigma}^0_2$-complete, by showing that player II has a winning strategy in the Wadge game $G_W(A, Q)$ for any $A \in \mathbf{\Sigma}^0_2(\mathcal{N})$.

74. For a property $P \subseteq \mathbb{N}$ of natural numbers, we use the following abbreviations:

$$\begin{aligned} \forall^\infty n P(n) \quad &:\Leftrightarrow \quad \{n \in \mathbb{N} : P(n)\} \text{ is cofinite} \quad \Leftrightarrow \quad \text{for large enough } n,\ P(n) \text{ holds} \\ \exists^\infty n P(n) \quad &:\Leftrightarrow \quad \{n \in \mathbb{N} : P(n)\} \text{ is infinite} \quad \Leftrightarrow \quad \text{for arbitrarily large } n,\ P(n) \text{ holds} \end{aligned}$$

Show that the set $Q_2 = \left\{x \in 2^{\mathbb{N}} : \forall^\infty n (x(n) = 0)\right\}$ is $\mathbf{\Sigma}^0_2$-complete and conclude that the set $N_2 = \left\{x \in 2^{\mathbb{N}} : \exists^\infty n (x(n) = 0)\right\}$ is $\mathbf{\Pi}^0_2$-complete.

75. Show that the following sets are $\mathbf{\Pi}^0_3$-complete:

(a) $P_3 = \left\{x \in 2^{\mathbb{N} \times \mathbb{N}} : \forall n \forall^\infty m (x(n, m) = 0)\right\}$,

HINT: Use Q_2 from the previous problem.

(b) $C_3 = \left\{x \in \mathbb{N}^{\mathbb{N}} : \lim_n x(n) = \infty\right\}$.

HINT: Reduce P_3 to C_3.

76. Each binary relation on $\mathbb{N}$ is an element of $\mathscr{P}(\mathbb{N}^2)$, which we may identify with $2^{\mathbb{N}^2}$. Thus, we can define

$$\begin{aligned} \mathrm{LO} &= \left\{x \in 2^{\mathbb{N}^2} : x \text{ is a linear ordering}\right\} \\ \mathrm{WO} &= \left\{x \in 2^{\mathbb{N}^2} : x \text{ is a well-ordering}\right\}. \end{aligned}$$

(a) Show that LO is a closed subset of $2^{\mathbb{N}^2}$ and that WO is co-analytic.

(b) Prove that WO is actually $\mathbf{\Pi}^1_1$-complete.

HINT: Define an appropriate ordering on a tree to show that $\mathrm{WF} \leq_W \mathrm{WO}$, where $\mathrm{WF} = \mathrm{Tr} \setminus \mathrm{IF}$.

77. Let $X_0 = \left\{x \in 2^{\mathbb{N}} : \forall^\infty n\ x(n) = 0\right\}$, $X_1 = \left\{x \in 2^{\mathbb{N}} : \forall^\infty n\ x(n) = 1\right\}$, and put $X = 2^{\mathbb{N}} \setminus (X_0 \cup X_1)$. Note that X_0 and X_1 are $\mathbb{E}_0$-classes, so all we did is throwing away from $2^{\mathbb{N}}$ two $\mathbb{E}_0$-classes. Define a continuous action of $\mathbb{Z}$ on X so that the induced orbit equivalence relation $E_{\mathbb{Z}}$ is exactly $\mathbb{E}_0|_X$.

78. Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ on a Polish space X. Show that there is a Borel equivariant[7] embedding $f : X \to (2^{\mathbb{N}})^{\Gamma}$, where $\Gamma \curvearrowright (2^{\mathbb{N}})^{\Gamma}$ by shift as follows: $\gamma \cdot y(\delta) = y(\delta\gamma)$, for $\gamma, \delta \in \Gamma$, $y \in (2^{\mathbb{N}})^{\Gamma}$. In particular, f is a Borel reduction of the induced orbit equivalence relations.

HINT: Let $(U_n)_n$ be a countable basis for X. For $x \in X$, to define $f(x)$, for each $\gamma \in \Gamma$ and $n \in \mathbb{N}$, record whether or not $\gamma \cdot x \in U_n$; this gives an element of $(2^{\mathbb{N}})^{\Gamma}$.

79. Show that $\mathbb{E}_v \sim_B \mathbb{E}_0$ by proving that $\mathbb{E}_v \sqsubseteq_B \mathbb{E}_0(\mathbb{N}) \sqsubseteq_c \mathbb{E}_0$ and $\mathbb{E}_0 \sqsubseteq_c \mathbb{E}_v$.

HINT: Use that each $x \in \mathbb{R}$ can be uniquely written as

$$x = \frac{a_1}{1!} + \frac{a_2}{2!} + \cdots + \frac{a_n}{n!} + \cdots ,$$

where $a_1 = \lfloor x \rfloor$, for each $n \geq 2$, $a_n \in \{0, 1, ..., n-1\}$ and $\exists^\infty n(a_n \neq n-1)$; the latter condition is to ensure uniqueness.

80. Let E be an equivalence relation on a Polish space X. Prove that $\mathrm{id}(2^{\mathbb{N}}) \leq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$.

81. Fill in the details in the proof of Mycielski's theorem; namely, given a meager equivalence relation E on a Polish space X, write $E = \bigcup_n F_n$, where F_n are increasing and nowhere dense, and construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}} \subseteq X$ of vanishing diameter (with respect to a fixed complete metric d for X) with the following properties:

(i) U_s is nonempty open, for each $s \in 2^{<\mathbb{N}}$;

(ii) $\overline{U_{s^\frown i}} \subseteq U_s$, for each $s \in 2^{<\mathbb{N}}$, $i \in \{0, 1\}$;

(iii) $(U_s \times U_t) \cap F_n = \emptyset$, for all distinct $s, t \in 2^n$ and $n \in \mathbb{N}$.

82. Let $(X, \mathcal{T})$ be a Polish space and let E be an equivalence relation on X. For a family $\mathcal{F}$ of subsets of X, we say that $\mathcal{F}$ *generates* E if

$$xEy \iff \forall A \in \mathcal{F}(x \in A \Leftrightarrow y \in A).$$

Prove that the following are equivalent:

(1) E is smooth;

(2) There is a Polish topology $\mathcal{T}_E \supseteq \mathcal{T}$ on X (and hence automatically $\mathcal{B}(\mathcal{T}_E) = \mathcal{B}(\mathcal{T})$) such that E is closed in $(X^2, \mathcal{T}_E^2)$.

CAUTION: It is easy to make E closed in X^2 by refining the topology of X^2, but here we have to refine the topology of X so that E becomes closed in X^2.

(3) E is generated by a countable Borel family $\mathcal{F} \subseteq \mathcal{B}(\mathcal{T})$.

HINT: For (1)⇒(2), consider a Borel function witnessing the smoothness of E and make it continuous. For (2)⇒(3), assuming that E is closed, write $X^2 \setminus E = \bigcup_n U_n \times V_n$, with U_n, V_n disjoint open, and note that the saturations $[U_n]_E$ and $[V_n]_E$ are disjoint analytic sets; separate them by an invariant Borel set.

[7]A map is called equivariant if it commutes with the action, i.e. $\gamma \cdot f(x) = f(\gamma \cdot x)$, for $x \in X$.

83. (Blackwell's theorem) Let X be a Polish space and E be an equivalence relation on X generated by a countable family $\{B_n\}_{n\in\mathbb{N}}$ of Borel sets. Prove that a Borel set $B \subseteq X$ is E-invariant iff it belongs to the σ-algebra generated by $\{B_n\}_{n\in\mathbb{N}}$.

HINT: For $\Rightarrow$ direction, consider the function $f : X \to 2^{\mathbb{N}}$ by $x \mapsto (x_n)_{n\in\mathbb{N}}$, where $x_n = 1 \Leftrightarrow x \in B_n$, and use Problem 60.

84. **Prisoners and hats** ($\mathbb{E}_0$ version). The purpose of this problem is to illustrate the non-smoothness of $\mathbb{E}_0$, more particularly, how having a selector for $\mathbb{E}_0$ (provided by AC) can cause crazy things.

Problem. ω-many prisoners were sentenced to death, but they could get out under one condition: on the day of the execution they will be lined up, i.e. enumerated $(p_n)_{n\in\omega}$, so that everybody can see everybody else (but not themselves). Each of the prisoners will have a red or blue hat put on him, but he won't be told which color it is (although he can see the colors of other prisoners' hats). On command, all the prisoners (at once) make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they all go home free; otherwise all of them are executed ☠. The good news is that the prisoners thought of a plan the day before the execution, and indeed, all but finitely many prisoners guessed correctly the next day, so everyone was saved. How did they do it?

85. For Polish spaces X, Y, a function $f : X \to Y$ is called *universally measurable* if the f-preimages of open sets in Y are universally measurable sets. Prove that the composition of two universally measurable functions is universally measurable.

86. For a Borel equivalence relation E, show that if there is universally measurable reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$, then E is smooth (i.e. there is a *Borel* reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$).

HINT: It's ok to use big theorems.

87. Let $S \subseteq 2^{<\mathbb{N}}$.

(a) If S contains at most one element of each length, then $\mathcal{G}_S$ is acyclic[8].

HINT: Suppose there is a cycle (with no repeating vertex) and consider the longest $s \in S$ associated with its edges.

(b) If S contains at least one element of each length, then $E_{\mathcal{G}_S} = \mathbb{E}_0$.

HINT: For each $n \in \mathbb{N}$, show by induction on n that for each $s, t \in 2^n$ and $x \in 2^{\mathbb{N}}$, there is a path in $\mathcal{G}_S$ from $s^\frown x$ to $t^\frown x$, i.e. $s^\frown x$ can be transformed to $t^\frown x$ by a series of appropriate bit flips.

88. **Prisoners and hats** ($\mathcal{G}_0$ version[9]). The purpose of this problem is to illustrate that for $S = \mathbb{N}^{<\mathbb{N}}$ (or any other $S \subseteq \mathbb{N}^{<\mathbb{N}}$, as long as $E_{\mathcal{G}_S} = \mathbb{E}_0$), $\mathcal{G}_S$ does not admit a "reasonable" 2-coloring as the one provided by AC can cause strange things[10].

[8] Here we treat $\mathcal{G}_S$ as an undirected graph, i.e. we consider edges (x, y) and (y, x) to be the same.

[9] The author is thankful to Dat P. Nguyen for coming up with this version of the problem.

[10] The solution to the version of this problem with finitely many prisoners is the same, and, although there is no need to use AC in this case, the outcome is still as strange.

Problem. ω-many prisoners were sentenced to death and were going to be executed as follows: on the day of the execution they will be lined up, i.e. enumerated $(p_n)_{n\in\omega}$, so that everybody can see everybody else (but not themselves). Each of the prisoners will have a red or blue hat put on him, but he won't be told which color it is (although he can see the colors of other prisoners' hats). On command, each prisoner (one-by-one, starting from p_0, then p_1, then p_2, etc.) makes a guess as to what color he thinks his hat is. Whoever guesses right, goes home free. The good news is that the prisoners thought of a plan the day before the execution, so that at most one prisoner dies. How did they do it?

89. Let $\mathcal{G}$ be a Borel graph on a Polish space $(X, \mathcal{T})$. For $A \subseteq X$, let

$$N_{\mathcal{G}}(A) = \{x \in X : \exists y \in A((x, y) \in \mathcal{G})\}$$

denote the set of $\mathcal{G}$-neighbors of vertices in A. If $A = \{x\}$, we just write $N_{\mathcal{G}}(x)$. Call $\mathcal{G}$ locally countable (resp. finite) if for every $x \in X$, $N(x)$ if countable (resp. finite).

(a) Suppose $\mathcal{G}$ is such that for every Borel $A \subseteq X$, $N_{\mathcal{G}}(A)$ is Borel, and prove that $\chi_{\mathcal{B}}(\mathcal{G}) \leq \aleph_0$ iff there is a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that for every $x \in X$, $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$.

Hint: For $\Leftarrow$, use the fact that $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$ is witnessed by a basic open set $U_n \in \mathcal{T}_0$.

(b) Conclude if $\mathcal{G}$ is locally finite, then $\chi_{\mathcal{B}}(\mathcal{G}) \leq \aleph_0$.

(c) Prove that if the maximum degree of $\mathcal{G}$ is $\leq d$ (i.e. $|N_{\mathcal{G}}(x)| \leq d$ for every $x \in X$), then $\chi_{\mathcal{B}}(\mathcal{G}) \leq d + 1$.

Hint: First prove this theorem for a finite graph so you can see the algorithm. Now for our Borel $\mathcal{G}$, use part (b) to partition $X = \bigsqcup_n A_n$ into $\mathcal{G}$-independent Borel sets and start running your coloring algorithm on A_0 until you've colored all the vertices connected to A_0. Then do the same thing with the left over vertices of A_1, then with A_2, and so on.

(d) Conclude that the Borel chromatic number of the graph induced by an irrational rotation of the unit circle is 3.

90. Show that there exists a *universal analytic equivalence relation*, i.e. an analytic equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_{\boldsymbol{\Sigma}}$.

Hint: Take a $\mathcal{C}$-universal set $U \subseteq \mathcal{C} \times \mathcal{N}^2$ for $\boldsymbol{\Sigma}^1_1(\mathcal{N}^2)$ and let $\tilde{U}$ be obtained from U by replacing the fibers U_x, $x \in \mathcal{C}$, with their symmetric and transitive closures, so that each fiber $\tilde{U}_x$ is an equivalence relation on $\mathcal{N}$. Now define an appropriate equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ on $\mathcal{C} \times \mathcal{N}$.

91. The goal of this problem is to show that there is a *universal countable Borel equivalence relation*, i.e. a countable Borel equivalence relation $\mathbb{E}_\infty$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_\infty$.

(a) Letting $\mathbb{F}_\omega$ be the free group on ω-many generators and Γ be any countable group, define a Borel reduction $\rho : (2^{\mathbb{N}})^\Gamma \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ of the orbit equivalence relation E_Γ to the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift actions $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$ and $\mathbb{F}_\omega \curvearrowright (2^{\mathbb{N}})^{\mathbb{F}_\omega}$, respectively.

HINT: Every countable group is a homomorphic image of $\mathbb{F}_\omega$.

(b) Using the Feldman-Moore theorem (stated below) in tandem with Problem 78, conclude that the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift action of $\mathbb{F}_\omega$ on $(2^{\mathbb{N}})^{\mathbb{F}_\omega}$ is a universal countable Borel equivalence relation.

Theorem (Feldman-Moore)**.** *Every countable Borel equivalence relation on a Polish space is induced as the orbit equivalence relation of a Borel action of a countable group.*