Math 294 Week 12

4/9/2019 or 4/11/2019

Today we'll talk more about set theory!

Definition. Given a set $\mathcal{X}$ whose elements are sets, we let $\bigcup \mathcal{X}$ be the union of all the elements of $\mathcal{X}$. In symbols, $x \in \mathcal{X}$ iff $x \in X$ for some $X \in \mathcal{X}$. For example, if $\mathcal{X} = \{(-3, 1), (0, 5]\}$, then $\bigcup \mathcal{X} = (-3, 1) \cup (0, 5] = (-3, 5]$.

Definition. A set X is called *transitive* if, whenever $y \in x \in X$, then $y \in X$. Notice the similarities to $<$ being a transitive relation on $\mathbb{R}$!

Theorem. X is transitive iff, for each $x \in X$, we also have $x \subseteq X$.

Theorem. $\varnothing$ is a transitive set.

Theorem. If X is a transitive set, then so is $X^+ = X \cup \{X\}$ (called the successor of X).

Theorem. If A and B are transitive sets, then so are $A \cap B$ and $A \cup B$.

Problem Set:

Problem 1. Prove the following:

- For any X, if X is transitive, then so is $\mathcal{P}(X)$.
- For any X, if X is transitive, then so is $\bigcup X$.
- Justify why $\bigcup\{\varnothing, \mathcal{P}(\varnothing), \mathcal{P}(\mathcal{P}(\varnothing)), \mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))), \dots\}$ is a transitive set.

Problem 2. Denote $\varnothing$ by 0. Define ω to be the following set:

$$\omega = \{0, 0^+, 0^{++}, 0^{+++}, \dots\}.$$

Prove the following:

- For shorthand, let 0^{+n} denote the element of ω which is followed by n-many $+$'s. Prove by induction on $n \in \mathbb{N}$ that 0^{+n} is transitive for each n.
- Justify (using the second bullet of problem 1) that ω is a transitive set.
- Prove that $\bigcup \omega = \omega$.

Problem 3. Let X be any set. X may or may not be transitive, but we'll show that every X is contained in a least transitive set $tc(X)$, called the transitive closure of X. For some additional notation, let's write $\bigcup_n A = \bigcup \cdots \bigcup A$, where the $\bigcup$ symbol occurs n times.

- For any X, if X is transitive, then so is $\bigcup X$. (Notice that this implies by induction that $\bigcup_n X$ will also be transitive.)
- Let $tc(X) = X \cup \bigcup_1 X \cup \bigcup_2 X \cup \cdots$. Intuitively, $x \in tc(X)$ if either x is an element of X, x is an element of an element of X, x is an element of an element of an element of X, etc. Use the first bullet of this problem to justify why $tc(X)$ is a transitive set.
- Prove that if Y is a transitive set and $X \subseteq Y$, then $tc(X) \subseteq Y$. (Hint, by induction show $\bigcup_n X \subseteq Y$.)