Math 294 Week 15

4/30/2019 or 5/1/2019

Today we'll talk about more about limits and how it relates to topology. Recall the following definition:

Definition. A sequence $\{a_n\}$ in $\mathbb{R}$ converges to a if for every $\varepsilon > 0$ there's an $N \in \mathbb{N}$ such that, for each $n \geq N$, $|a_n - a| < \varepsilon$. This is written $a_n \to a$.

An incredibly important notion in topology is that of a closed set:

Definition. A subset $F \subseteq \mathbb{R}$ is closed if, whenever $x_n \to x$ and $x_n \in F$ for each n, then $x \in F$. In words, a set is closed if it contains all of its limit points.

Definition. A set $G \subseteq \mathbb{R}$ is open if G^c is closed.

Problem 1. Prove the following:

1. If F_1 and F_2 are closed, then $F_1 \cup F_2$ is also a closed set.
2. Give an example of infinitely many closed sets $F_1, F_2, \ldots, F_n, \ldots$ such that $F_1 \cup F_2 \cup \cdots$ is not closed. (So we can't generalize the previous proof to infinite unions.)

Problem 2. Prove that if $\{F_i : i \in I\}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed. Recall that $x \in \bigcap_{i \in I} F_i$ iff for each $i \in I$, $x \in F_i$.

Problem 3. Let $F \subseteq \mathbb{R}$ be any subset. Prove the following:

1. If F is closed, then for each $x \notin F$ there's an open interval $x \in (a, b)$ such that $(a, b) \subseteq F^c$.
2. Assume that for each $x \notin F$ there's an open interval $x \in (a, b)$ such that $(a, b) \subseteq F^c$. Prove that F is closed.