Math 294 Week 3

1/29/2019 or 1/31/2019

This worksheet will emphasize function composition and its relationship with injective/surjective functions discussed last week. As a reminder from last week, recall the following definitions:

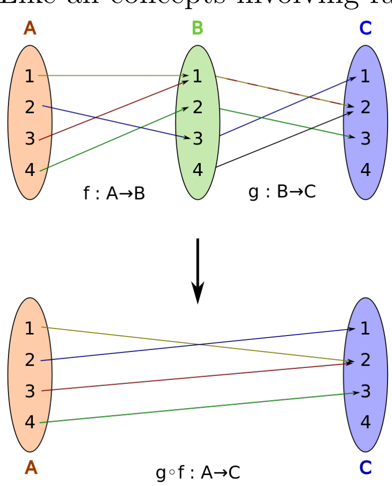
Definition. A function $f\colon A \to B$ is:

1. injective (or 1-1) if for all $a, a' \in A$, $a \neq a'$ implies that $f(a) \neq f(a')$;
2. surjective (or onto) if for all $b \in B$ there's an $a \in A$ such that $f(a) = b$;
3. bijective if it's both injective and surjective.

If we have two functions, say f and g, it's natural to ask if there is a way to combine the two functions together. Depending on the context, there can be different ways to do this, but common way to do this is to *compose* f and g together:

Definition. Let functions $f\colon A \to B$ and $g\colon B \to C$ where the codomain of f equals the domain of g. The composition of f and g is defined to be the following function, denoted $g \circ f$. Given $a \in A$, then we define $g \circ f(a) = g(f(a))$

Like all concepts involving functions, we may think of it as a picture!



Theorem 1. Let $f\colon A \to B$ and $g\colon B \to C$ be functions.

1. If f and g are both injective, then so is $g \circ f$.
2. If f and g are both surjective, then so is $g \circ f$.
3. If f and g are both bijective, then so is $g \circ f$.

Before I list the problems, Let's collect two more definitions that will be helpful in solving some of the problems:

Definition. Given two functions f and g with the same domain A, then we say that f equals g, written $f = g$, if for any $a \in A$, $f(a) = g(a)$.

Definition. Given any set A, we may define a particular function, called the identity function. The function is denoted $id_A\colon A \to A$, and is defined by $id_A(a) = a$ for every $a \in A$.

Here are the problems to work on this week. You must submit one of the problems with the word "prove" in it (problems 1-5) by the end of class to be marked present for this week.

Something to think about. For the definition $f = g$, why do you think we require the domain of f and g to be the same but not the codomain? Also, is id_A a bijection?

Problem 1. Prove: if $f\colon A \to B$ is an injective function and A is nonempty (i.e. has at least one element), then there's a surjection $g\colon B \to A$ such that $g \circ f = id_A$. (g is called a left inverse.)

Problem 2. Prove: if $f\colon A \to B$ is a surjective function and B is nonempty (i.e. has at least one element), then there's an injection $g\colon B \to A$ such that $f \circ g = id_B$. (g is called a right inverse.)

Problem 3. Let $f\colon A \to B$ and $g\colon B \to A$ be functions. Prove the following: if $g \circ f = id_A$, then f is injective and g is surjective. Explain why this is the same as proving the following statement: "if if $f \circ g = id_B$, then g is injective and f is surjective."

Problem 4. If $f\colon A \to B$ is 1-1 and $g\colon B \to C$ is onto, then is $g \circ f$ necessarily 1-1? onto? For both questions, either prove or disprove give a counterexample.

Problem 5. If $f\colon A \to B$ is onto and $g\colon B \to C$ is 1-1, then is $g \circ f$ necessarily 1-1? onto? For both questions, either prove or disprove give a counterexample.